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On the generalized eigenfunctions system of the Sturm–Liouville problem

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Abstract. In this paper we investigate eigenfunctions and generalized eigenfunctions system of the Sturm–Liouville problem with classical boundary condition on the left boundary and nonlocal boundary conditions of four types on the right boundary.

Keywords: Sturm–Liouville problem, generalized eigenfunctions, nonlocal boundary conditions.

Let us analyze the Sturm–Liouville problem

$$-u'' = \lambda u, \qquad t \in (0, 1) \tag{1}$$

with one classical boundary condtition

$$u(0) = 0, \tag{2}$$

and other nonlocal two-point boundary condition of Samarskii–Bitsadze type:

$$u'(1) = \gamma u(\xi), \qquad \text{(Case 1)} \tag{3_1}$$

$$u'(1) = \gamma u'(\xi), \qquad \text{(Case 2)} \tag{3_2}$$

$$u(1) = \gamma u'(\xi), \qquad \text{(Case 3)} \tag{3_3}$$

$$u(1) = \gamma u(\xi), \qquad \text{(Case 4)} \tag{3_4}$$

with the parameters $\gamma \in \mathbb{R}$ and $\xi \in [0, 1]$. Also we analyze the Sturm–Liouville problem (1) with boundary condition

$$u'(0) = 0 \tag{4}$$

on the left side and with nonlocal boundary conditions (3) on the right side of the interval, we enumerate these cases from Case $1'$ till Case $4'$ accordingly. We denote problems (1)–(2) in the case nonlocal boundary conditions (3_1)–(3_4) as P1, P2, P3, P4 and problems (1), (4) in the case nonlocal boundary conditions (3_1)–(3_4) as P1$'$, P2$'$, P3$'$, P4$'$, respectively. Note that the index in references denotes the case. If there are no index then the rule (or results) hold on in all cases of nonlocal boundary conditions. The dependence of the such Sturm–Liouville problem spectrum on nonlocal boundary conditions parameters is analyzed in [1–4].

Remark 1. [Classical case]. We have classical case for $\gamma = 0$. Eigenvalues in this case are well known:

$$\lambda_k^{\mathrm{cl}} = k^2\pi^2, \qquad u_k^{\mathrm{cl}}(t) = \sin(k\pi t), \qquad k \in \mathbb{N}; \tag{$5_{3,4}$}$$

$$\lambda_k^{\mathrm{cl}} = (k-1/2)^2\pi^2, \qquad u_k^{\mathrm{cl}}(t) = \sin((k-1/2)\pi t), \qquad k \in \mathbb{N}; \tag{$5_{1,2}$}$$

$$\lambda_k^{\mathrm{cl}} = (k-1/2)^2\pi^2, \qquad u_k^{\mathrm{cl}}(t) = \cos((k-1/2)\pi t), \qquad k \in \mathbb{N}; \tag{$5_{3',4'}$}$$

$$\lambda_k^{\mathrm{cl}} = (k-1)^2\pi^2, \qquad u_k^{\mathrm{cl}}(t) = \cos((k-1)\pi t), \qquad k \in \mathbb{N}. \tag{$5_{1',2'}$}$$

The same case we get for $\xi = 0$ (Problems P1, P4, P2′, P3′), $\xi = 1$ and $\gamma \neq 1$ (Problems P2, P4, P2′, P4′). In the case $\xi = 1$ and $\gamma = 1$ (Problems P2, P4, P2′, P4′) we have generate case with one left boundary condition. So, we omit these cases and define $D_\xi := [0,1]$ (Problems P3, P1′), $D_\xi := (0,1]$ (Problems P1, P3′), $D_\xi := [0,1)$ (Problems P2, P4′), $D_\xi := (0,1)$ (Problems P4, P2′).

The problem (1)–(2) has the solution $U(t) = t$ for $\lambda = 0$ and solution $U(t) = \sin(qt)$ for $\lambda \neq 0$. The problem (1), (3) has the solution $U(t) = 1$ for $\lambda = 0$ and solution $U(t) = \cos(qt)$ for $\lambda \neq 0$. Note, that $CU(t)$ will be nontrivial solution for all $C \neq 0$, too. Parameter $q \in \{z \in \mathbb{C} : -\pi/2 < \arg z \leqslant \pi/2\}$ and $\lambda = q^2$. Further in this paper we take q only in the rays $q = x \geqslant 0$, $q = -\mathrm{i}x$, $x \leqslant 0$ instead of $q \in \mathbb{C}$. So, we investigate only real eigenvalues λ. We can find q as γ-values of characteristic functions.

Let us write expression of characteristic function in each case of nonlocal boundary condition [1,2,4] for $\xi \in D_\xi$:

$$\gamma = \frac{1}{\xi}\frac{f(x)}{g(\xi x)}, \qquad \begin{cases} f(x) := \cosh x, & g(x) := \frac{\sinh x}{x} \ \text{ for } x \leqslant 0,\\ f(x) := \cos x, & g(x) := \frac{\sinh x}{x} \ \text{ for } x \geqslant 0; \end{cases} \tag{6_1}$$

$$\gamma = \frac{f(x)}{g(\xi x)}, \qquad \begin{cases} f(x) := \cosh x, & g(x) := \cosh x \ \text{ for } x \leqslant 0,\\ f(x) := \cos x, & g(x) := \cos x \ \text{ for } x \geqslant 0; \end{cases} \tag{$6_{2,4'}$}$$

$$\gamma = \frac{f(x)}{g(\xi x)}, \qquad \begin{cases} f(x) := \frac{\sinh x}{x}, & g(x) := \cosh x \ \text{ for } x \leqslant 0,\\ f(x) := \frac{\sin x}{x}, & g(x) := \cos x \ \text{ for } x \geqslant 0; \end{cases} \tag{6_3}$$

$$\gamma = \frac{1}{\xi}\frac{f(x)}{g(\xi x)}, \qquad \begin{cases} f(x) := \frac{\sinh x}{x}, & g(x) := \frac{\sinh x}{x} \ \text{ for } x \leqslant 0,\\ f(x) := \frac{\sin x}{x}, & g(x) := \frac{\sin x}{x} \ \text{ for } x \geqslant 0; \end{cases} \tag{$6_{4,2'}$}$$

$$\gamma = \frac{f(x)}{g(\xi x)}, \qquad \begin{cases} f(x) := x\sinh x, & g(x) := \cosh x \ \text{ for } x \leqslant 0,\\ f(x) := -x\sin x, & g(x) := \cos x \ \text{ for } x \geqslant 0; \end{cases} \tag{$6_{1'}$}$$

$$\gamma = \xi\frac{f(x)}{g(\xi x)}, \qquad \begin{cases} f(x) := \cosh x, & g(x) := x\sinh x \ \text{ for } x < 0,\\ f(x) := \cos x, & g(x) := -x\sin x \ \text{ for } x > 0. \end{cases} \tag{$6_{3'}$}$$

We formulate obvious properties of the functions f and g as following proposition. Some of these properties were investigated in [1,2,4].

Proposition 1. The point $z_0 = 0$ is zero of the second order for the function f in Case $1'$ and for the function g in Case $3'$:

$$f(z_0) = f'(z_0) = 0,\ \ f''(z_0) \neq 0, \tag{$7_{1'}$}$$

$$g(z_0) = g'(z_0) = 0,\ \ g''(z_0) \neq 0. \tag{$7_{3'}$}$$

Another zeroes points z of the functions $f(x)$, $g(x)$ are zeroes of the first order:

$$f(z) = 0,\ \ f'(z) \neq 0, \quad g(z) = 0,\ \ g'(z) \neq 0. \tag{8}$$

These positive zeroes of the first order of the function f are equal:

$$z_k := (k - 1/2)\pi, \quad k \in \mathbb{N}, \tag{$9_{1,2,3',4'}$}$$

$$z_k := k\pi, \quad k \in \mathbb{N}; \tag{$9_{3,4,1',2'}$}$$

positive zeroes of the first order of the function g are equal:

$$\wp_k := (k - 1/2)\pi, \quad k \in \mathbb{N}, \tag{$10_{2,3,1',4'}$}$$

$$\wp_k := k\pi, \quad k \in \mathbb{N}. \tag{$10_{1,4,2',3'}$}$$

The characteristic function has zero point z, if $f(z) = 0$ and $g(z\xi) \neq 0$ (Problems P1–P4, P1$'$–P4$'$). For characteristic functions (6) we have next zeroes points of the function f:

$$z_k = (k - 1)\pi, \quad k \in \mathbb{N}, \tag{$11_{1'}$}$$

$$z_k = (k - 1/2)\pi, \quad k \in \mathbb{N}, \tag{$11_{1,2,3',4'}$}$$

$$z_k = k\pi, \quad k \in \mathbb{N}. \tag{$11_{3,4,2'}$}$$

Note that zeroes points are the same for all ξ and they are on the vertical lines in the domain $D_{x,\xi} := \mathbb{R} \times D_\xi$. The point $x = 0$ is zero point only for Problem P1$'$ and it is zero of the second order for all $\xi \in [0, 1]$.

Characteristic function (Problems P1–P4, P1$'$–P4$'$, $\xi \neq 0$) has pole point $\wp$ if $g(\wp) = 0$ and $f(\wp/\xi) \neq 0$. For characteristic functions (6) we have next zero points $\wp$ for function g:

$$\wp_k = (k - 1/2)\pi, \quad k \in \mathbb{N}, \tag{$12_{2,3,1',4'}$}$$

$$\wp_k = k\pi, \quad k \in \mathbb{N}, \tag{$12_{1,4,2'}$}$$

$$\wp_k = (k - 1)\pi, \quad k \in \mathbb{N}. \tag{$12_{3'}$}$$

For these cases the poles of the characteristic function are p_k and $\wp_k = \xi p_k$. So, the poles are on the hyperbolae $\xi x = \wp_k$ in the domain $D_{x,\xi}$. Point $x = 0$ is pole point only for Problem P3$'$, and it is the pole of the second order for all $\xi \in (0, 1]$ and in this case hyperbola degenerates to line $x = 0$. Characteristic function (Problems P2, P3, P1$'$, P4$'$) for $\xi = 0$ is entire function, i.e., there is no poles points.

Remark 2. All positive zeroes and positive poles for these problems are the first order. If for some ξ we have $f(z_k) = g(p_l) = 0$, $k, l \in \mathbb{N}$ then this point $z_k = p_l = c$ is constant eigenvalue point. We have $\gamma(c) \neq 0$ for problems P1–P4 and P1′–P4′. Geometrically we get constant eigenvalues points as intersection vertical zeroes lines and poles hyperbolae in the domain $D_{x,\xi}$. We note that $x = 0$ is not constant eigenvalue point.

The point $x_{cr} \in \mathbb{R}$ is a *critical point* of real characteristic function, if $\gamma'(x_{cr}) = 0$. Critical points of the characteristic function are maximum and minimum points of this function. For problems P1–P4 and P1′–P4′ there exist infinitely many positive critical points. For problem P3, there exists also a negative critical point if $\xi \in (\sqrt{3}/3, 1)$ and for problem P3′ negative critical point exists for all $\xi \in (0, 1)$ [1,2]. We have $\gamma''(x_{cr}) \neq 0$ for problems P1–P4 and P1′–P4′.

Critical points, zeroes, poles of the characteristic function and constant eigenvalues points are important for the investigation of multiple eigenvalues and generalized eigenfunctions [4].

DEFINITION 1. *A point is called* double point *if it is critical point of the characteristic function and not a constant eigenvalue point; or it is constant eigenvalue point and not a pole point of the characteristic function. A critical point of the characteristic function is called* triple point *if it is also constant eigenvalue point.*

Double points exist for all investigated problems, triple points exist only for problems P2, P2′, P4, P4′ [1–4].

If we want to analyze first order generalized functions, we need to consider additional differential problem

$$-v'' - \lambda(x)v = U(t), \quad t \in (0, 1) \tag{13}$$

with one of the two boundary conditions:

$$v(0) = 0 \tag{14}$$

or

$$v'(0) = 0; \tag{15}$$

and with other nonlocal boundary condition

$$v'(1) - \gamma(x)v(\xi) = 0, \tag{$16_{1,1'}$}$$

$$v'(1) - \gamma(x)v'(\xi) = 0, \tag{$16_{2,2'}$}$$

$$v(1) - \gamma(x)v'(\xi) = 0, \tag{$16_{3,3'}$}$$

$$v(1) - \gamma(x)v(\xi) = 0; \tag{$16_{4,4'}$}$$

where $U(t)$ is an eigenfunction for a real eigenvalue point x and $\gamma(x) = \pm x^2$. A solution of problems (13)–(14), (16) and (13), (15)–(16) is called a generalized eigenfunction of the first order. If generalized eigenfunctions exist, then the eigenvalue λ is a multiple eigenvalue.

LEMMA 1. *A generalized eigenfunctions v of the first order exist at the positive double points, and $v = C_1 U(t) + V(t)$, where*

$$V(t) := \frac{t\cos(xt)}{2x}, \tag{$17_{1,2,3,4}$}$$

$$V(t) := -\frac{t\sin(xt)}{2x}. \tag{$17_{1',2',3',4'}$}$$

In Case 3 and 3′ generalized eigenfunction v of the first order exists also at the negative double points and $v = CU(t) + V(t)$, where

$$V(t) := \frac{t\cosh(xt)}{2x}, \tag{18_3}$$

$$V(t) := -\frac{t\sinh(xt)}{2x}. \tag{$18_{3'}$}$$

Remark 3. We can take on right of the equation (13) function $CU(t)$ instead $U(t)$. In this case $v = C_1 U(t) + CV(t)$.

Let us analyze second order generalized eigenfunctions. We consider additional differential problem

$$-w'' - \lambda(x)w = V(t), \quad t \in (0,1) \tag{19}$$

with one of the two boundary conditions:

$$w(0) = 0 \tag{20}$$

or

$$w'(0) = 0; \tag{21}$$

and with other nonlocal boundary condition

$$w'(1) - \gamma(x)w'(\xi) = 0, \tag{$22_{2,2'}$}$$

$$w(1) - \gamma(x)w(\xi) = 0; \tag{$22_{4,4'}$}$$

where $V(t)$ is generalized eigenfunction of the first order (17) or (18).

LEMMA 2. *A generalized eigenfunction w of the second order exists only at the triple points, and $w = CU(t) + W(t)$, where*

$$W(t) := -\frac{t\cos(xt) + t^2x\sin(xt)}{8x^3}, \tag{$23_{2,2'}$}$$

$$W(t) := \frac{t\sin(xt) - t^2x\cos(xt)}{8x^3}. \tag{$23_{4,4'}$}$$

Remark 4. We can take on right hand side of the equation (19) function $C_1U(t) + CV(t)$ instead $V(t)$. In this case $w = C_2U(t) + C_1V(t) + CW(t)$.

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REZIUMĖ

S. Pečiulytė, A. Štikonas. Apie Šturmo ir Liuvilio uždavinio prijungtinių funkcijų sistemos

Šiame darbe nagrinėjamas Šturmo ir Liuvilio uždavinys su klasikine sąlyga kairiajame krašte ir keturių tipų nelokaliosiomis dvitaškėmis kraštinėmis sąlygomis dešiniajame krašte. Straipsnyje suformuluotos lemos apie pirmosios ir antrosios eilės prijungtinių funkcijų egzistavimą. Surastos šios funkcijos.

Raktiniai žodžiai: Šturmo ir Liuvilio uždavinys, prijungtinės funkcijos, nelokaliosios kraštinės sąlygos.