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A labeled sequent calculus for propositional linear time logic

Romas Alonderis

Institute of Mathematics and Informatics, Vilnius University
Akademijos 4, LT-08663 Vilnius
E-mail: romas.alonderis@mii.vu.lt

Abstract. A labeled sequent calculus **LSC** for propositional linear discrete time logic **PLTL** is introduced. Its sub-calculus $\mathbf{LSC}^{-}_{\mathbf{TL}}$ is proved to be complete for some class of **PLTL** sequents.

Keywords: labeled sequent calculus, temporal logic.

1 Introduction

Temporal logic is a special type of modal logic. It provides a formal system for qualitatively describing and reasoning about how the truth values of assertions change over time. Propositional linear discrete time logic **PLTL** with temporal operators "next" and "always" is considered in the present paper.

Various syntactical proof-search systems are used for **PLTL**. Some of them are:

- Sequent calculi with the invariant rule

$$\frac{\Gamma \to \Delta, I; I \to \circ I; I \to A}{\Gamma \to \Delta, \Box A}\ (\to \Box_{\mathrm{I}}),$$

  [10, 11]. There are some interesting works in which invariant-free (and cut-free) calculi for PLTL are constructed [3, 6].

- Sequent calculi with the infinitary rule

$$\frac{\Gamma \to \Delta, A; \Gamma \to \Delta, \circ A; \ldots; \Gamma \to \Delta, \overbrace{\circ \ldots \circ}^{n} A; \ldots}{\Gamma \to \Delta, \Box A}\ (\to \Box_{\omega}),$$

  [12]. There are some interesting works concerning finitization of ω-type rule $(\to \Box_{\omega})$ (see, e.g., [4]).

- Proof procedures containing loop-type axioms for logics sub-logic of which is propositional temporal branching time logic [9].

- Labeled sequent calculi [1, 2].

- Resolution-type proof procedures based on formulas in some normal form, see, e.g., [5].

In the present paper a labeled sequent calculus **LSC** is presented. Its sub-calculus $\mathbf{LSC}^{-}_{\mathbf{TL}}$ is proved to be complete for some easily defined but large class of **PLTL** sequents. Unlike the other deductive systems mentioned above, calculi **LSC** and $\mathbf{LSC}^{-}_{\mathbf{TL}}$ are loop-axiom and invariant and infinitary rule free, which allows to construct effective proof-search procedures based on the calculi.

2 Syntax

Formulas are defined in the traditional way.

Formulas of the shape $x^k : A$, where $k \in \{0\} \cup N$ (in particular, $x^0 = x$) and A is a formula, are called labeled formulas, l-formulas for short; x is called a label or a variable and k its power. Labels/variables are denoted by u, x, y, z, w and the corresponding powered labels by u^k, x^k, y^k, z^k, and w^k. The intended meaning of '$x : A$' is "A holds at some moment of time x" and that one of '$x^k : A$' is "A holds at the k-th from x moment of time".

Expressions $m^n : A$, where $m, n \in \{0\} \cup N$, are called fixed-label formulas and 'm^n' fixed labels.

One more type of formulas is $x^m \leqslant y^m$, where $m \geqslant 0$. Such formulas are called order atoms.

Sequents are objects of the type $\Gamma \to \Delta$, where Γ and Δ are some finite multisets of formulas.

Labeled sequents, l-sequents for short, are objects of the type $\Gamma \to \Delta$, where Γ is some finite multiset of labeled formulas and order atoms; the same for Δ except that order atoms do not occur in it.

3 Semantics

Kripke semantics of **PLTL** is defined as follows.

$(\{0\} \cup N \times \mathbf{P}) \overset{\tau}{\mapsto} \{\top, \bot\}$, where $\mathbf{P}$ is the set of propositional variables.

$(\{0\} \cup N \times \mathbf{F}) \overset{\phi}{\mapsto} \{\top, \bot\}$, where $\mathbf{F}$ is the set of formulas and ϕ is defined in the following way.

1. $\phi(i, E) = \tau(i, E)$, where E is an atomic formula;
2. $\phi(i, A)$ is defined in the common way if A is of the shape $\neg B$ or $B\theta C$, where θ is a logical connective;
3. $\phi(i, \circ A) = \top$ iff $\phi(i + 1, A) = \top$; otherwise, $\phi(i, \circ A) = \bot$;
4. $\phi(i, \Box A) = \top$ iff $\phi(j, A) = \top$ for all j such that $j \geqslant i$; otherwise, $\phi(i, \Box A) = \bot$.

Some more notation:

(1) $(i^k : A) = \phi(i + k, A)$;

(2) $\models i^k : A$ iff $(i^k : A) = \top$ for any ϕ;

(3) $\models x^k : A$ iff $\models i^k : A$ for all $i \geqslant 0$;

(4) $\models A$ iff $\phi(i, A) = \top$ for all $i \geqslant 0$ and every ϕ

here A is a label free formula, $k \geqslant 0$, and '$\models$' denotes validity.

$\mathcal{L} \overset{\nu}{\mapsto} (\{0\} \cup N)$, where $\mathcal{L}$ is the set of labels.

The stable sequent S_ν is obtained from S by substituting every label x_i by $\nu(x_i)$.

$\mathbf{S}_\nu \overset{\varsigma}{\mapsto} \{\top, \bot\}$, where $\mathbf{S}_\nu$ is the class of stable sequents and ς is defined as follows: if

$$S = x_1^{i_1} : A_1, \dots, x_k^{i_k} : A_k \to x_{k+1}^{i_{k+1}} : B_{k+1}, \dots, x_{k+m}^{i_{k+m}} : B_{k+m},$$

then $\varsigma(S_\nu) = \top$, iff there are ϕ, ν, and t, where $1 \leqslant t \leqslant (k+m)$, such that $(\nu(x_t)^{i_t} : A_t) = \bot$ or $(\nu(x_t)^{i_t} : B_t) = \top$. Otherwise, $\varsigma(S_\nu) = \bot$.

A stable sequent S_ν is valid, denoted by $\models S_\nu$, iff $\varsigma(S_\nu) = \top$ for any τ.

A stable sequent S_ν is an axiom if it is of the shape $\Gamma, l^k : E \to m^n : E, \Delta$, where $l + k = m + n$.

A labeled sequent S is valid, $\models S$ in notation, iff every stable sequent obtained from S is valid.

4 Labeled sequent calculi LSC and $\mathbf{LSC}^-_{\mathbf{TL}}$

The labeled sequent calculus **LSC** for **PLTL** is defined as follows:

1. Axioms:

$$\Gamma, x^k : E \to x^k : E, \Delta,$$

   where E is an atomic formula.

2. Logical rules:

$$\frac{x^k : A, x^k : B, \Gamma \to \Delta}{x^k : A \wedge B, \Gamma \to \Delta}\ (\wedge \to), \qquad \frac{\Gamma \to x^k : A, \Delta; \quad \Gamma \to x^k : B, \Delta}{\Gamma \to x^k : A \wedge B, \Delta}\ (\to \wedge),$$

$$\frac{x^k : A, \Gamma \to \Delta; \quad x^k : B, \Gamma \to \Delta}{x^k : A \vee B, \Gamma \to \Delta}\ (\vee \to), \qquad \frac{\Gamma \to x^k : A, x^k : B, \Delta}{\Gamma \to x^k : A \vee B, \Delta}\ (\to \vee),$$

$$\frac{\Gamma \to x^k : A, \Delta}{x^k : \neg A, \Gamma \to \Delta}\ (\neg \to), \qquad \frac{\Gamma, x^k : A \to \Delta}{\Gamma \to x^k : \neg A, \Delta}\ (\to \neg),$$

$$\frac{\Gamma \to x^k : A, \Delta; \quad x^k : B, \Gamma \to \Delta}{x^k : A \supset B, \Gamma \to \Delta}\ (\supset\to), \qquad \frac{\Gamma, x^k : A \to x^k : B, \Delta}{\Gamma \to x^k : A \supset B, \Delta}\ (\to\supset).$$

   Here A and B arbitrary formulas.

3. Temporal rules:

$$\frac{\Gamma \to x^{k+1} : A, \Delta}{\Gamma \to x^k : \circ A, \Delta}\ (\to \circ), \qquad \frac{x^{k+1} : A, \Gamma \to \Delta}{x^k : \circ A, \Gamma \to \Delta}\ (\circ \to),$$

$$\frac{x \leqslant y, \Gamma \to y^k : A, \Delta}{\Gamma \to x^k : \Box A, \Delta}\ (\to \Box),$$

$$\frac{y^{k+m} : A, x^{k+m} \leqslant y^{k+m}, x^k : \Box A, \Gamma \to \Delta}{x^{k+m} \leqslant y^{k+m}, x^k : \Box A, \Gamma \to \Delta}\ (\Box \to).$$

   Here $k, m \geqslant 0$; y in $(\to \Box)$ does not occur in the conclusion.

4. Rules for order atoms:

$$\frac{x \leqslant x, \Gamma \to \Delta}{\Gamma \to \Delta}\,\text{Ref}, \qquad \frac{x^{k+1} \leqslant y^{k+1}, x^k \leqslant y^k, \Gamma \to \Delta}{x^k \leqslant y^k, \Gamma \to \Delta}\,\text{Fwd}^{+1},$$

$$\frac{x \leqslant z, x \leqslant y, y \leqslant z, \Gamma \to \Delta}{x \leqslant y, y \leqslant z, \Gamma \to \Delta}\,\text{Trans},$$

$$\frac{y \leqslant z, x \leqslant y, x \leqslant z, \Gamma \to \Delta; \quad z \leqslant y, x \leqslant y, x \leqslant z, \Gamma \to \Delta}{x \leqslant y, x \leqslant z, \Gamma \to \Delta}\,\text{Lin}.$$

Here x, y, z are unequal in pairs in Trans and Lin; $x \leqslant x$ does not occur in Γ in Ref; $x^{k+1} \leqslant y^{k+1}$ does not occur in Γ in Fwd; $x \leqslant z$ does not occur in Γ in Trans; In Lin, neither $y \leqslant z$ nor $z \leqslant y$ occur in Γ neither can be obtained by some backward applications of Trans.

The calculus $\mathbf{LSC^-_{TL}}$ is obtained from **LSC** by dropping Trans and Lin.

A formula F is called derivable in the labeled sequent calculus **LSC** ($\mathbf{LSC^-_{TL}}$) iff $LSC(LSC^-_{TL}) \vdash \to x : F$.

A sequent S is called derivable in **LSC** ($\mathbf{LSC^-_{TL}}$) iff $LSC(LSC^-_{TL}) \vdash x : S$.

The Hilbert-style calculus **HSC** for **PLTL** is defined by axioms:

A_0: propositional tautologies; A_1: $\circ\neg p \equiv \neg\circ p$;

A_2: $\circ(p \supset q) \supset (\circ p \supset \circ q)$; A_3: $\Box(p \supset q) \supset (\Box p \supset \Box q)$;

A_4: $\Box p \supset p$; A_5: $\Box p \supset \circ\Box p$; A_6: $p \wedge \Box(p \supset \circ p) \supset \Box p$;

and derivation rules:

$$\frac{p}{\circ p}\circ, \qquad \frac{p}{\Box p}\Box, \qquad \frac{p, p \supset q}{q}mp,$$

where p and q are arbitrary **PLTL** formulas. It is well known that this calculus is sound and complete for **PLTL**, see, e.g. [7].

5 Some Properties of LSC and $\mathbf{LSC^-_{TL}}$

Lemma 1. *If $LSC(LSC^-_{TL}) \vdash^V S$, then $LSC(LSC^-_{TL}) \vdash^{V'} S(w/u)$ and $h(V') \leqslant h(V)$, where $S(w/u)$ is obtained from S by substituting the label w for the label u.*

A rule is height-preserving admissible if, whenever its premiss(es) is (are) derivable, also its conclusion is derivable with the same bound on the derivation height.

Lemma 2. *The rule of weakening*

$$\frac{\Gamma \to \Delta}{\Gamma', \Gamma \to \Delta, \Delta'}\ (w)$$

is height-preserving admissible in **LSC** *and* $\mathbf{LSC^-_{TL}}$.

A rule is height-preserving invertible if, whenever its conclusion is derivable, also its premiss(es) is (are) derivable with the same bound on the derivation height.

Lemma 3. *All* **LSC** *rules are height-preserving invertible in* **LSC**, *and all* $\mathbf{LSC^-_{TL}}$ *rules are height-preserving invertible in* $\mathbf{LSC^-_{TL}}$.

Lemma 4. *The rules of contraction*

$$\frac{C, C, \Gamma \to \Delta}{C, \Gamma \to \Delta}\ (c \to) \quad \text{and} \quad \frac{\Gamma \to \Delta, C, C}{\Gamma \to \Delta, C}\ (\to c)$$

are height-preserving admissible in **LSC** *and* $\mathbf{LSC}^-_{\mathbf{TL}}$.

A sequent S is called proper if the fact that $x^k \leqslant y^k$ occurs in S, where $k \geqslant 0$, implies that $x^0 \leqslant y^0$ occurs in S.

Theorem 1. *The rule of cut*

$$\frac{\Pi \to C, \Lambda; C, \Gamma \to \Delta}{\Pi, \Gamma \to \Lambda, \Delta}\ cut$$

is admissible in **LSC** *and* $\mathbf{LSC}^-_{\mathbf{TL}}$, *where the premisses are proper.*

Lemma 5. *All* **LSC** *rules are correct: if the premise(s) is (are) valid, then so is the conclusion. In addition, if the conclusion is valid, then so is (are) the premise(s).*

Lemma 6. *Any labeled sequent of the shape* $\Gamma, x : A \to x : A, \Delta$ *is derivable in* **LSC** *and* $\mathbf{LSC}^-_{\mathbf{TL}}$.

Theorem 2. $HSC \vdash^d F$ *implies* $LSC^-_{TL} \vdash \to x : F$, *where the induction axiom* A_6 *is not used in* d.

If $\Gamma = A_1, \dots, A_n$, then $\theta\Gamma = (A_1\theta \dots \theta A_n)$, where $\theta \in \{\wedge, \vee\}$. If $S = \Gamma \to \Delta$, then $F(S) = \neg(\wedge\Gamma) \vee (\vee\Delta)$.

By Theorem 2 and invertibility of the rules $(\to \vee)$, $(\neg \to)$, and $(\wedge \to)$, $\mathbf{LSC}^-_{\mathbf{TL}}$ is complete for sequents $S = \Gamma \to \Delta$ such that $F(S)$ is derivable in **HSC** without using the axiom A_6.

An example of non-derivable in $\mathbf{LSC}^-_{\mathbf{TL}}$formula is

$$\Box A \supset \Box\Box A.$$

Theorem 2 implies that this formula is not derivable in **HSC** without the axiom A_6. This formula is derivable in **LSC**.

Some examples of non-derivable in **LSC** formulas are

$$(A \wedge \circ\Box A) \supset \Box A \quad \text{and} \quad \big(A \wedge \Box(A \supset \circ A)\big) \supset \Box A.$$

We get by Theorem 2 that these formulas are not derivable in **HSC** without A_6.

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REZIUMĖ

Žymėtas sekvencinis skaičiavimas propozicinei tiesinio laiko logikai

R. Alonderis

Darbe yra pateiktas žymėtas sekvencinis skaičiavimas propozicinei tiesinio laiko logikai. Įrodyta, kad šis skaičiavimas yra pilnas tam tikros nagrinėjamos logikos sekvencijų klasės atžvilgiu.

Raktiniai žodžiai: žymėtas sekvencinis skaičiavimas, laiko logika.