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Sequent systems for PLTL

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Abstract. We consider three sequent calculi for propositional linear temporal logic (**PLTL**) which allow us to formalize the properties of operator "always".

The main new results presented in the paper are: (1) introduction of the calculus with looping axioms; (2) the direct proof that the presented calculi are equivalent; (3) the proof of completeness of the calculi with looping axioms and with invariant-like rule based on completeness of the calculus with the infinitary ω-type rule.

Keywords: temporal logics, sequent calculi, ω-type rule, weak-induction rule, looping axioms, invariant-like rule.

1 Introduction

In this paper, we consider propositional linear temporal logic (**PLTL**) with temporal operators "next" and "always". These operators allow us to explain in simple and evident way that combinations of them requires use of induction-like tools.

We consider only sequent-like calculi for **PLTL**. Such calculi allow us (1) to present in evident way induction-like properties of operators "next" and "always"; (2) to construct sequent calculi with the nice constructive properties. For this reason we do not consider calculi of the type [3] since this calculus destroys sub-formula property, introducing the new operator "unless".

2 Description of language and infinitary calculus for PLTL

The language of considered PLTL contains a set of propositional symbols $P, P_1, P_2, \ldots,$ $Q, Q_1.Q_2, \ldots$, the set of logical connectives $\supset, \wedge, \vee, \neg$, temporal operators $\Box$ ("always") and $\circ$ ("next"). The language does not contain the temporal operator $\diamond$ ("sometimes"), assuming that $\diamond A = \neg\Box\neg A$. We assume that time is linear, discrete, and ranges over the set of natural numbers.

Formulas are constructed in the traditional way from propositional symbols, using the logical connectives and temporal operators. The formula $\circ A$ means "A is true at the next moment of time"; the formula $\Box A$ means "A is true now and in all moments of time in the future".

We consider Gentzen-type calculi, which are based on sequents, i. e., the formal expressions $\Gamma \to \Delta$, where Γ and Δ are final multisets of formulas.

The infinitary calculus $\mathbf{G}_\omega\mathbf{T}$ is defined by the following postulates:

1. Axioms: $\Gamma, A \to \Delta, A$;

2. Traditional logical rules;

3. Temporal rules:

$$\frac{\Gamma \to \Delta}{\Pi, \circ\Gamma \to \Theta, \circ\Delta}\ (\circ), \qquad \frac{A, \circ\Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (\Box \to),$$

$$\frac{\Gamma \to \Delta, A;\ \Gamma \to \Delta, \circ A;\ \dots\ \Gamma \to \Delta, \overbrace{\circ \dots \circ}^{n} A;\ \dots}{\Gamma \to \Delta, \Box A}\ (\to \Box_\omega).$$

Here: A and B denote arbitrary formulas; $\Gamma, \Delta, \Pi, \Theta$ denote finite, possibly empty, multisets of formulas.

It follows from [10] that $\mathbf{G}_\omega\mathbf{T}$ is sound and complete for **PLTL**.

There are some interesting works concerning finitization of ω-type rule $(\to \Box_\omega)$ (see, e.g., [1]).

Derivation in $\mathbf{G}_\omega\mathbf{T}$ are built in the form of infinite trees each branch of which is finitary. The height of a derivation D (denoted by $O(D)$) is evaluated in ordinal.

A derivation containing some application of the rule $(\to \Box_\omega)$ is called informal.

A $\mathbf{G}_\omega\mathbf{T}$ derivation D is called atomic if every axiom occurring in D is of the type $\Gamma, P \to \Delta, P$, where P is a propositional variable.

Lemma 1. *An arbitrary $\mathbf{G}_\omega\mathbf{T}$ derivation can be transformed into an atomic one.*

It is easy to see that all rules of $\mathbf{G}_\omega\mathbf{T}$, except the rule $(\circ)$, are invertible.

We further present a specialization of the rule $(\circ)$, obtaining the invertible rule $(\circ)'$.

A sequent S is a primary one iff $S = \Sigma_1, \circ\Gamma_1 \to \circ\Gamma_2, \Sigma_2$, where Σ_i $(i \in \{1, 2\})$ is empty or consists of propositional symbols; $\circ\Gamma_i$ $(i \in \{1, 2\})$ is empty or consists of formulas of the type $\circ A$, where A is arbitrary.

Lemma 2. *By backward application of $\mathbf{G}_\omega\mathbf{T}$ rules, except the rule $(\circ)$, any sequent S can be reduced to a (infinite) set of primary sequents $\Psi = S_1, \dots, S_n, \dots$ such that if $G_\omega T \vdash S$, then $G_\omega T \vdash S_i$, for any sequent $S_i \in \Psi$. Here S is the root and the sequents in Ψ are the leaves of a $\mathbf{G}_\omega\mathbf{T}$ backward proof-search tree.*

Let $\mathbf{G}'_\omega\mathbf{T}$ be the calculus obtained from $\mathbf{G}_\omega\mathbf{T}$ by replacing the rule $(\circ)$ by the following one

$$\frac{\Pi \to \Delta}{\Sigma_1, \circ\Pi \to \Sigma_2, \circ\Delta}\ (\circ)',$$

where the conclusion is primary and $\Sigma_1 \cap \Sigma_2 = \emptyset$. It is obvious that the rule $(\circ)'$ is invertible. It follows from Lemma 2 that $\mathbf{G}_\omega\mathbf{T}$ and $\mathbf{G}'_\omega\mathbf{T}$ are equivalent.

3 Finitary calculi $\mathbf{G_I T}$ and $\mathbf{G_L T}$

The infinitary calculi $\mathbf{G}_\omega\mathbf{T}$ and $\mathbf{G}'_\omega\mathbf{T}$ posses several nice properties, for example:

(1) they allow us to present simple and evident completeness proof (see, e.g., [10]);

(2) all rules of $\mathbf{G}'_\omega\mathbf{T}$ are invertible.

Despite of these properties these calculi have the serious shortcoming that derivations containing the infinitary rule $(\to \Box_\omega)$ are informal. To avoid this shortcoming, several finitary calculi can be presented.

3.1 Calculus with invariant-like rule

The calculus $\mathbf{G_I T}$ ($\mathbf{G'_I T}$) is obtained from the calculus $\mathbf{G}_\omega\mathbf{T}$ ($\mathbf{G}'_\omega\mathbf{T}$, correspondingly) by:

(1) Assuming that F in the axioms is an atomic formula or $F = \Box F'$.

(2) Replacing the infinitary rule $(\to \Box_\omega)$ by the rule

$$\frac{\Gamma \to \Delta, I;\ I \to \circ I;\ I \to A}{\Gamma \to \Delta, \Box A}\ (\to \Box_{\mathrm{I}}),$$

where the formula I (called an invariant formula) is constructed from subformulas of formulas in the conclusion of rule $(\to \Box_{\mathrm{I}})$. There are some works in which some constructive methods for finding invariant formulas are presented, e.g., [7]. It follows [12] (see also [6]) that the calculus $\mathbf{G_I T}$ is sound and complete for **PLTL**.

Remark 1. The sequent $\Box P \to \Box P$ is a simple example showing that $\mathbf{G_I T}$ does not posses the atomic derivation property.

3.2 Calculus with loop-type axioms

The calculus $\mathbf{G_L T}$ ($\mathbf{G'_L T}$) is obtained from the calculus $\mathbf{G}_\omega\mathbf{T}$ ($\mathbf{G}'_\omega\mathbf{T}$, correspondingly) by:

(a) replacing the infinitary rule $(\to \Box_\omega)$ by the weak-induction rule

$$\frac{\Gamma \to \Delta, A;\ \Gamma \to \Delta, \circ\Box A}{\Gamma \to \Delta, \Box A}\ (\to \Box_{\mathrm{L}})$$

and

(b) adding loop-type (or looping) axioms: a sequent S' is a loop-type (or looping) axiom, iff (1) S' is above a sequent S on a branch of a derivation tree, (2) S is such that it subsumes S' ($S \succcurlyeq S'$ in notation), i.e., S' can be obtained from S by using the structural rule of weakening, and (3) there is the right premise of $(\to \Box_{\mathrm{L}})$ between S and S', and there is not the left premiss of $(\to \Box_{\mathrm{L}})$ between S and S'.

Analogously as in [5], it can be proved that $\mathbf{G_L T}$ ($\mathbf{G'_L T}$) is sound and complete for **PLTL**.

Defining that $\circ\Box A$ is a sub-formula of $\Box A$ and that $\circ\Box A$ and $\Box A$ have the same complexity, we get that all rules of the calculi $\mathbf{G_L T}$ and $\mathbf{G'_L T}$ have the sub-formula property, and complexity of any premiss of any rule is not greater than that of the conclusion.

All rules of $\mathbf{G'_L T}$ are invertible.

We do not consider the invariant-free sequent calculus of [3], since this calculus does not preserves sub-formula property, introducing the new operator "until" ("unless").

We do not consider tableaux-like calculi, see, e.g., [13, 9].

We do not consider resolution-like calculi, see, e.g., [2].

From soundness and completeness of the infinitary calculus $\mathbf{G}_\omega\mathbf{T}$, invariant-like calculus $\mathbf{G_I T}$, and loop-type calculus $\mathbf{G_L T}$, we get that all the three calculi are equivalent. In the next section we present the direct proof that the considered calculi are equivalent.

4 Direct proof of equivalence of the calculi $\mathbf{G}_\omega\mathbf{T}$, $\mathbf{G}_\mathbf{I}\mathbf{T}$, and $\mathbf{G}_\mathbf{L}\mathbf{T}$

First we prove

Lemma 3. *If $G_LT \vdash S$, then $G_IT \vdash S$, where S is an arbitrary sequent.*

Lemma 4. *The rule*

$$\frac{S_1 = \Gamma \to \Delta, A;\ S_2 = A, \Pi \to \Lambda}{S^* = \Gamma, \Pi \to \Delta, \Lambda}\ (cut)$$

is admissible in $\mathbf{G}_\mathbf{L}\mathbf{T}$.

Lemma 5. *For each $n \geqslant 1$, it is true that $G_LT \vdash S_n^* = \circ\Box A \to \overbrace{\circ \cdots \circ}^{n} A$.*

Lemma 6. *If $G_LT \vdash S$, then $G_\omega T \vdash S$, where S is an arbitrary sequent.*

To prove that a sequent is derivable in $\mathbf{G}_\mathbf{I}\mathbf{T}$ if it is derivable in $\mathbf{G}_\omega\mathbf{T}$ we extend the notion of a sequent allowing infinitary sequents. An infinitary sequent can contain not only finitary but also infinitary formulas. Infinitary sequents are often studied (with various notions of infinintary) in modal logics (see, e.g., [8, 11]).

Let $\mathbf{G}_\mathbf{I}^\infty\mathbf{T}$ b×e the calculus obtained from $\mathbf{G}_\mathbf{I}\mathbf{T}$ by:

(1) extending the language by infinitary conjunction and infinitary sequents and
(2) adding the infinitary rules

$$\frac{A_0, \ldots, A_n, \ldots, \Gamma \to \Delta}{\wedge_{i=0}^{\infty} A_i, \Gamma \to \Delta}\ (\wedge_\infty \to)$$

and

$$\frac{\Gamma \to A_0, \Delta;\ \ldots\ \Gamma \to A_n, \Delta;\ \ldots}{\Gamma \to \wedge_{i=0}^{\infty} A_i, \Delta}\ (\to \wedge_\infty).$$

Lemma 7. *If $G_\omega T \vdash S$, then $G_I^\infty T \vdash S$ for an arbitrary sequent S.*

Lemma 8. *If $G_\omega T \vdash S$, then $G_IT \vdash S$ for an arbitrary sequent S.*

Proposition 1. *The (cut) rule is admissible in* $\mathbf{G}_\omega\mathbf{T}$.

Lemma 9. *If $G_\omega T \vdash S_1 = I \to \circ I$ and $G_\omega T \vdash S_2 = I \to A$, then $G_\omega T \vdash S_3 = I \to \circ^n A$ for each $n \geqslant 0$ and any formulas I and A.*

Lemma 10. *If $G_IT \vdash S$, then $G_\omega T \vdash S$ for an arbitrary sequent S.*

To prove that an arbitrary sequent is derivable in $\mathbf{G}_\mathbf{L}\mathbf{T}$ if it is derivable in $\mathbf{G}_\mathbf{I}\mathbf{T}$, we introduce the semi-Hilbert type calculus $\mathbf{HG}_\mathbf{I}\mathbf{T}$, which is obtained from $\mathbf{G}_\mathbf{I}\mathbf{T}$ by replacing the rule $(\to \Box_\mathrm{I})$ by the induction axiom

$$A, \Box(A \supset \circ A) \to \Box A$$

and adding the (cut) rule. Completeness of $\mathbf{HG}_\mathbf{I}\mathbf{T}$ is proved almost in the same way as completeness of the Hilbert type version of $\mathbf{G}_\mathbf{I}\mathbf{T}$ (see, e.g., [4]).

Lemma 11. *If $HG_IT \vdash S$, then $G_LT \vdash S$, where S is an arbitrary sequent.*

Lemma 12. *If $G_IT \vdash S$, then $G_LT \vdash S$, where S is an arbitrary sequent.*

Proof. The proof of the lemma follows from completeness of $\mathbf{HG_IT}$ and Lemma 11.

Theorem 1. *The calculi $\mathbf{G_\omega T}$, $\mathbf{G_I T}$, and $\mathbf{G_L T}$ are equivalent.*

Proof. The proof of the theorem follows from Lemmas 3, 6, 8, 10, and 12.

Theorem 2. *The calculi $\mathbf{G_L T}$ and $\mathbf{G_I T}$ are complete for* **PLTL**.

Proof. The proof follows from Theorem 1 and completeness of the calculus $\mathbf{G_\omega T}$.

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REZIUMĖ

Sekvencinės sistemos PTL logikai

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Straipsnyje nagrinėjami trys sekvenciniai skaičiavimai skirti propozicinei tiesinio laiko logikai. Sintaksiniu būdu įrodytas šių skaičiavimų ekvivalentumas.

Raktiniai žodžiai: laiko logika, sekvenciniai skaičiavimai, ω taisyklė, silpnos indukcijos taisyklė, ciklinės aksiomos, invariantinė taisyklė.