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# Saturation method for reflexive common knowledge logic

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**Abstract.** This paper discusses the use of saturation procedure in order to check looping sequents in reflexive common knowledge logic. Traditional approach states that common knowledge operator is defined by some induction-like axiom and requires the use of some looping sequents. The loopcheck-free saturation-like procedure lets us obtain special loop-free sequents.

**Keywords:** saturation method, common knowledge logic, reflexive common knowledge logic.

## Introduction

A reflexive common knowledge logic (RCL) containing individual knowledge operators (satisfying multi-modal logic $K_n$), reflexive "common knowledge" and "everyone knows" operators is considered. The common knowledge operator is defined by some induction-like axiom. To manage with this axiom some looping sequents (acting as non-logical axioms) are used. These looping sequents create some additional technological difficulties. To avoid these difficulties some loopcheck-free saturation-like procedure involving marked modal rules for "common knowledge" operator is considered. This saturation procedure terminates when special sequents are obtained.

## 1 Description of language and initial calculus

The language of considered RCL contains a set of propositional symbols $P_1, P_2, \dots$ $Q, Q_1, Q_2, \dots$; the set of logical connectives $\supset$, $\wedge$, $\vee$, $\neg$; finite set of agent constants $i, i_1, i_2, \dots$; multi-agent knowledge modality $K(i)$, where $i$ is agent constant; everyone knows operator $E$, common knowledge operator $C$.

Formulas in the considered calculi are constructed in the traditional way from propositional symbols using the logical symbols and knowledge operators.

The formula $K(i)A$ means "agent $i$ knows $A$". The operator $K(i)A$ behaves as modality of modality of logic $K$. The formula $E(A)$ means "every agent knows $A$", i.e. $E(A) \equiv \bigwedge_{i=1}^{n} K(i)A$ ($n$ is a number of agents). The formula $C(A)$ means "$A$ is common knowledge of all agents"; it is assumed that there is perfect communication between agents. The formula $C(A)$ has the same meaning as the infinite formula $\bigwedge_{k \geqslant 0} E^k(A)$, where $E^k(A) = A$ if $k = 0$ and $E^k(A) = E^{k-1}(E(A))$, if $k > 0$. The operators $C$ and $E$ behave as modalities of logic S5. In addition these operators have the following powerful properties $C(A) \equiv A \wedge E(C(A))$ (fixed point) and $A \wedge C(A \supset$

$E(A)) \supset C(A)$ (induction). Formal semantics of the formulas $K(i)$, $E(A)$, $C(A)$ are defined as in the reflexive common knowledge logic [3].

Along with formulas we consider sequents, i.e. formal expressions $A_1, \ldots, A_k \to B_1, \ldots B_m$, where $A_1, \ldots, A_k$ $(B_1, \ldots B_m)$ is a multiset of formulas. The sequent is interpreted as the formula $\bigwedge_{i=1}^{k} A_i \supset \bigvee_{j=1}^{m} B_j$, $k \geqslant 0$, $m \geqslant 0$. Below we present two types of sequent calculi for RCL.

Looping type calculus LRC is defined by following postulates.

*Axioms*:

Logical axiom: $\Gamma, A \to \Delta, A$.

Loop-type axiom which is defined by the following way: a sequent $S$ is a loop-type axiom if (1) $S$ is above a sequent $S'$ on a branch of a derivation tree; (2) $S$ subsumes $S'$ ($S \geqslant S'$ in notation) i.e. $S'$ can be obtained from $S$ using weakening and contraction only ($S$ and $S'$ coincide in separate case); (3) there is right premise of $(\to C)$ (see below) between $S$ and $S'$.

Logical rules consist of traditional invertible rules for logical symbols.

Modal rules:

$$\frac{A, E(\mathrm{C}(A)), \Gamma \to \Delta}{\mathrm{C}(A), \Gamma \to \Delta}\ (C \to),$$

$$\frac{\Gamma \to \Delta, A; \Gamma \to \Delta, E(\mathrm{C}(A))}{\Gamma \to \Delta, \mathrm{C}(A)}\ (\to C),$$

$$\frac{\Gamma \to \Delta, \bigwedge_{i=1}^{m} K_i(A)}{\Gamma \to \Delta, E(A)}\ (\to E),$$

$$\frac{\bigwedge_{i=1}^{m} K_i(A), \Gamma \to \Delta}{E(A), \Gamma \to \Delta}\ (E \to),$$

$$\frac{\Gamma \to A}{\Pi, K_i\Gamma \to \Delta, K_i(A)}\ (K_i).$$

$m$ – number of agents.

*Example 1.* Let $S = P$, $C(P \supset E(P)) \to C(P)$ and let us construct a derivation of $S$ in LRC:

$$\cfrac{\cfrac{\cfrac{S_1^* = P_1 C(P \supset E(P) \to C(P)}{P \ldots \to P;\ P,\ E(P),\ E(C(P \supset E(P))) \to E(C(P))}\ (E)}{P \to P;\ E(P),\ E(C(P \supset E(P))) \to C(P)}\ (\supset\to)}{S = P,\ C(P \supset E(P)) \to C(P)}\ (\to C)$$

It is obvious that $S_1$ is looping axiom ($S = S_1$). Other top sequents are logical axioms, therefore LRC $\vdash$S. In the presented derivation the following rule was used (admissible in LRC):

$$\frac{\Gamma \to A}{\Pi, E(\Gamma) \to \Delta, E(A)}(E).$$

Infinitary calculus $\mathrm{R}_{\Omega}$CA is obtained from LRC by (1) dropping the looping axioms and (2) replacing the rule $(\to C)$ by the following infinitary rule:

$$\frac{\Gamma \to \Delta, A;\ \Gamma \to \Delta, E(A);\ \ldots;\ \Gamma \to \Delta, E^k A;\ \ldots}{\Gamma \to \Delta, \mathrm{C}(A)}\ (\to C\omega)$$

where $E^0(A) = A$; $E^k(A) = E(E^{k-1}(A))$, $k \geqslant 1$.

## 2 Description of saturated calculus

We introduce fined set of $\delta$ ($\delta \in \{*, +, \ldots\}$) which are applied to any formula and define as follows:

1. $(P)^\delta = P^\delta$; $\quad (P^\delta)^\delta = P^\delta$;
2. $(A \odot B)^\delta = A^\delta \odot B^\delta$, $\quad \odot \in \{\supset, \wedge, \vee\}$;
3. $(GA)^\delta = GA^\delta$, $\quad G \in \{\neg, K_i, E\}$;
4. $(C(A))^\delta = C(A^\delta)$ if $A \neq A^\delta$ and $C^\delta(A)$ if $A = A^\delta$.

A marked formula of the shape $A^\delta$ ($\delta \in \{*, +, \ldots\}$) is called $\delta$-marked. A sequent $S$ is $\delta$-reduced ($\delta \in \{*, +, \ldots\}$) if $S = \sum_1^\delta E(\prod_{11}^\delta)K(\prod_{12}^\delta)C^\delta(\Gamma^\delta) \to \sum_2^\delta E(\prod_{21}^\delta)K(\prod_{22}^\delta)$, $C^\delta(\Delta^\delta)$, where $\sum_i^\delta$ ($i \in \{1, 2\}$) consists of marked propositional symbols; $G\Pi_{i2}^\delta$ ($i \in \{1, 2\}$) is a multiset of marked formulas of the shape $G_i A^\delta$.

A $\delta$-reduced ($\delta \in \{*, +, \ldots\}$) sequent is proper (improper) if $C^\delta(\Delta^\delta) \neq \emptyset$ ($C^\delta(\Delta^\delta) = \emptyset$ correspondingly).

A $\delta$-reduced proper sequent is positive ($p$-final, in short).

Let us introduce $\delta$-marked ($\delta \in \{*, +, \ldots\}$) rules which allow us (along with other rules) to generate in backward way $\delta$-reduced sequents:

$$\frac{A^\delta, E(C^\delta(A^\delta)), \Gamma \to \Delta}{C(A), \Gamma \to \Delta}\ (C^\delta \to),$$

$$\frac{\Gamma \to \Delta, A^\delta;\ \Gamma \to \Delta, E(C^\delta(A^\delta))}{\Gamma \to \Delta, C(A)}\ (\to C^\delta),$$

$$\frac{\Pi_1^\delta, \Gamma_1^\delta \to A^\delta}{\Pi, G(\Pi_1^\delta), G(\Gamma_1^\delta) \to \Delta, G(A^{\delta_0})}\ (G^\delta)$$

$G \in \{E, K_i\}$, $\delta_0 \in \{\emptyset, \delta\}$, $\delta \in \{*, +, \ldots\}$.

A saturated calculus SRC is obtained from $\mathrm{R}_\Omega\mathrm{C}$ by adding (as non-logical axioms) $p$-final sequents and replacing the rules $(C \to)$, $(\to C)$, $(E)$ with the rules $(C^\delta \to)$, $(\to C^\delta)$, $(G^\delta)$, $G \in \{E, K_i\}$.

Let us construct a derivation of the sequent from the Example 1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\begin{array}{c} P^+, C^+(P^+ \supset E(P^+)) \to C^+(P^+) \\ \vdots\ (D) \\ P^*, C^*(P^* \supset E(P^*)) \to C^*(P^*) \end{array}}{P \to P^*; P,\ E(P^*), E(C^*(P^* \supset E(P^*))) \to E(C^*(P^*))}\ (E^*)}{P \to P^*; P,\ E(P^*), E(C^*(P^* \supset E(P^*))) \to C(P)}\ (\to C^*)}{P,\ (P^* \supset E(P^*)),\ E(C^*(P^* \supset E(P^*))) \to C(P)}\ (\supset\to)}{P, C(P \supset E(P)) \to C(P)}\ (C^* \to)$$

A derivation $(D)$ consists of backward applications of the rules $(C^+ \to)$, $(\supset\to)$, $(\to C^+)$, $(E^\delta)$. It is easy to see that using $\delta$-marks ($\delta \in \{*, +, \ldots\}$) it is possible to define a complexity $|S|$ of a sequent $S$ in such way that $|S_i| < |S|$ where $|S_i|$ is a complexity of a premise of a rule $(i)$ and $|S|$ is a complexity of the conclusion of rule $(i)$. Using this property we can get a decidability of described saturated calculus.

It should be stressed that fined of set marks are need for verifying a derivability of sequents in the presented saturated calculus. Let $G^*RC$ be a calculus obtained from the basic saturated calculus removing from the language the mark $+$, i.e., the non-logical axioms are only arbitrary $*$-reduced sequents. Let $S$ be the sequent $\rightarrow P$, $C(\neg C(P))$. It is easy to prove that $S^*$ is not provable in SRC but $G^*RC \vdash S$. Therefore the saturated calculus with only one mark is not correct, i.e. we can derive invalid sequents.

*Remark 1.* Let $S^{**}RC$ be a calculus obtained from $SRC$ replacing the rule 4 of marking the modality C by the following rule:

$$4^\delta.\bigl(C(A)\bigr)^\delta = C^\delta\bigl(A^\delta\bigr)$$

$\delta \in \{*, +, \ldots\}$. Then the calculus $S^{**}RC$ is incomplete. Indeed, for example, the $SRC \vdash S = C(A) \rightarrow C(C(\vee B))$ but it is not derivable in $S^{**}RC$.

Necessity to have marked rules $(K_i^\delta)$ and $(E^\delta)$ $(\delta \in \{*, +, \ldots\})$ can be demonstrated with the help of valid sequents $C(P) \rightarrow K_i(P)$ and $C(P) \rightarrow E(P)$, correspondingly.

We can prove that the looping calculus, infinitary calculus and saturated calculus are equivalent. From this fact we can get that presented saturated calculus $SRC$ is sound and complete.

## References


[1] P. Aleate and R. Gore. Cut-free single pass tableaux for the logic of common knowledge. In *Proc. of Workshop on Agents and Deduction at Tableaux 2007*, 2007.

[2] K. Brunlez and T. Studez. Symantic cut-elimination for common knowledge. In *Proc. of Methods for Modalities, vol.* 5, 2007.

[3] J.J.Ch. Meyer and W. von der Hoek. *Epistemie Logic for AI and Computer Science.* Cambridge University Press, Cambridge, 1995.


REZIUMĖ

**Prisotinimo metodas bendrojo žinojimo logikai**

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Straipsnyje pateikiamas prisotinimo metodo naudojimas bendrojo žinojimo logikoje siekiant patikrinti ciklinius sekventus refleksyviojoje bendrojo žinojimo logikoje. Tradcininio požiūrio tailymas leidžia daryti prielaidą, kad bendrojo žinojimo operatorius apibrėžiamas naudojant indukcines aksiomas ir reikalauja ciklinių aksiomų naudojimo. Beciklis prisotinimo metodas leidžia gauti specialius beciklius sekventus.

*Raktiniai žodžiai*: prisotinimo metodas, bendrojo žinojimo logika, refleksyvi bendro žinojimo logika.