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Quaternionic Bézier curves, surfaces and volume

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Abstract. We extended the rational Bézier construction for linear, bi-linear and three-linear map, by allowing quaternion weights. These objects are Möbius invariant and have halved degree with respect to the real parametrization. In general, these parametrizations are in four dimensional space. We analyse when a special the three-linear parametrized volume is in usual three dimensional subspace and gives three orthogonal family of Dupine cyclides.

Keywords: circle, Dupine cyclide, inversion, quaternion.

Introduction

A linear rational Bézier curve with quaternionic control points and weights is a circle (see [3]). A bi-linear rational Bézier surface with quaternion control points and weights is a cyclide (see [2]). In this note we define and analyse a special tri-linear quaternionic rational map. This could be useful in geometric modelling and discreet differential geometry (see [1]).

1 Notations and definitions

We denote by $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ the set of real numbers, complex numbers and quaternion numbers respectively.

In general, the quaternion set $\mathbb{H}$ can be represented as

$$\mathbb{H} = \bigl\{ q = [r, p\,] \bigm| r \in \mathbb{R},\ p \in \mathbb{R}^3 \bigr\} = \mathbb{R}^4 \tag{1}$$

We denote real and imaginary parts of quaternion $q = [r, p]$ by $\mathrm{Re}(q) = r$, $\mathrm{Im}(q) = p$. The multiplication in the algebra $\mathbb{H}$ is defined as

$$[r_1, p_1][r_2, p_2] = [r_1 r_2 - p_1 \cdot p_2,\ r_1 p_2 + r_2 p_1 + p_1 \times p_2], \tag{2}$$

where $p_1 \cdot p_2, p_1 \times p_2$ are scalar and vector products in $\mathbb{R}^3$. We denote by $\bar{q} = [r, -p]$ a conjugate quaternion to $q = [r, p]$, $|q| = \sqrt{r^2 + p \cdot p} = \sqrt{q\bar{q}}$ is the length of the quaternion, $q^{-1} = \bar{q}/|q|^2 = [r/|q|^2, -p/|q|^2]$ denote the multiplicative inverse of q, i.e. $qq^{-1} = q^{-1}q = 1$. Denote the set of pure imaginary quaternions

$$\mathrm{Im}(\mathbb{H}) = \bigl\{ [0, p\,] \bigm| p \in \mathbb{R}^3 \bigr\} = \mathbb{R}^3. \tag{3}$$

2 Quaternion rational linear Bézier curve

We represent a curve in Bézier form with quaternionic control points $a_k \in \operatorname{Im}\mathbb{H}$ and weights $w_k \in \mathbb{H}$, $k = 0, 1$. Formally speaking, we are dealing with a quaternion function in homogeneous coordinates $(a_k w_k, w_k) \in \mathbb{H}^2$, $k = 0, 1$. The *quaternion rational linear Bézier curve* is defined in as customary quotient:

$$c_h(t) = n(t)d(t)^{-1}, \quad \text{where} \tag{4}$$
$$n(t) = a_0 w_0(1-t) + a_1 w_1 t, \qquad d(t) = w_0(1-t) + w_1 t, \tag{5}$$
$$h = \{h_k = (a_k, w_k) \in \mathbb{H}^2,\ k = 0, \ldots, 1\}. \tag{6}$$

Here we consider $n(t)$, $d(t)$ as quaternions, $d(t)^{-1}$ is an inverse quaternion and $n(t)d(t)^{-1}$ is the multiplication of two quaternions. So, in general, we have $c_h(t) \in \mathbb{H} = \mathbb{R}^4$.

Remark 1. If we change the weights w_0, w_1 to $1, w_1 w_0^{-1}$ the parametrized curve $c_h(t)$ is the same. Moreover, if we change the parameter t to $s = \rho t/(1 - t + \rho t)$ ($\rho \in \mathbb{R}$) and weights w_0, w_1 to $w_0, w_1/\rho$ then the curve is the same too.

A quaternionic rational Bézier curve $c(t)$ of degree one is a circular arc with two endpoints a_0, a_1. This case is well understood (see [3]). We note that these curves are invariant with respect to Möbius transformation.

3 Bi-linear quaternion Bézier surfaces

In this section we consider a special bi-linear quaternionic Bézier surface. We remind how to construct a principal Dupin cyclide patch. For detailed description we recommended the preprint [4].

Let us consider four points $Q = \{a_0, a_1, a_3, a_2\}$ cyclically arranged on a circle and an unit orthogonal frame of two vectors $\{v_1, v_2\}$ at the point a_0. We will use notations:

$$\delta_{i,j} = |a_i - a_j|, \quad i, j = 0, 1, 2, 3,\ i \neq j, \tag{7}$$
$$d_{i,j} = (a_i - a_j)/\delta_{i,j}, \quad i, j = 0, 1, 2, 3,\ i \neq j,$$
$$d_{i,j,k,\ldots,l,m} = d_{i,j} d_{j,k} \ldots d_{l,m}. \tag{8}$$

Using the triple data $T = \{Q, v_1, v_2\}$ we compute the following weights

$$w_0 = 1, \qquad w_1 = d_{0,1} v_1, \qquad w_2 = d_{0,2} v_2, \qquad w_3 = \frac{\delta_{1,2}}{\delta_{0,3}} d_{3,1,0} v_1 v_2. \tag{9}$$

Let denote by $H(T) = \{(a_i, w_i),\ i = 0, 1, 2, 3\}$ a collection of points and weights. We define a special bi-linear quaternion surface

$$D_{H(T)}(s,t) = n(s,t)d(s,t)^{-1}, \quad \text{where} \tag{10}$$
$$n(s,t) = a_0 w_0(1-s)(1-t) + a_1 w_1 s(1-t) + a_2 w_2(1-s)t + a_3 w_3 st,$$
$$d(s,t) = w_0(1-s)(1-t) + w_1 s(1-t) + w_2(1-s)t + w_3 st.$$

One can prove that such that the surface patch $D_{H(T)}(s,t)$ is in $\mathbb{R}^3$ (see [4]). Moreover, this patch is on some Dupine cyclide. We notice that the weights computed by formulas (9) are not unique. If we change weights (w_0, w_1, w_2, w_3) with weights $(w_0 q, \lambda w_1 q, \mu w_2 q, \lambda\mu w_3 q)$, $q \in \mathbb{H}$, $\lambda, \mu \in \mathbb{R}$ we get the same patch with different parametrization (see [4]).

4 Volume

Firstly, we call an elementary hexahedron $H = \{a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}, a_{123}\}$ a *spherical cube* if the following quadrangles

$$Q_3 = \{a_0, a_1, a_{12}, a_2\}, \qquad Q_2 = \{a_0, a_1, a_{13}, a_3\}, \qquad Q_1 = \{a_0, a_2, a_{23}, a_3\},$$
$$Q_3^* = \{a_{123}, a_{23}, a_3, a_{13}\}, \qquad Q_2^* = \{a_{123}, a_{23}, a_2, a_{12}\}, \qquad Q_1^* = \{a_{123}, a_{13}, a_1, a_{12}\}$$

are circular. According to classical Miquel theorem (see [1]) the point a_{123} is uniquely determined by the seven points $\{a_0, a_1, a_2, a_3, a_{12}, a_{13}, a_{23}\}$. In fact, we have $a_{123} = c_1 \cap c_2 \cap c_3$, where c_i is a circle through three points $\{a_i, a_{ij}, a_{ik}\}$, $i = 1, 2, 3$. Moreover, the hexahedron H is inscribed in a unique sphere.

Let us choose orthogonal unit frame $\{v_{01}, v_{02}, v_{03}\}$ at the point a_0. We consider three triples

$$T_3 = \{Q_3, v_{01}, v_{02}\}, \qquad T_2 = \{Q_2, v_{01}, v_{03}\}, \qquad T_1 = \{Q_1, v_{02}, v_{03}\}.$$

For all triples T_i, $i = 1, 2, 3$ we compute the weights $\{w_0, w_1, w_2, w_3, w_{12}, w_{13}, w_{23}\}$ by the formulas (9) and associate three Dupin ciclides $D_{H(T_1)}$, $D_{H(T_2)}$, $D_{H(T_3)}$ as we explained in the previous section.

We define a weight w_{123} by the formula

$$w_{123} = \frac{\delta_{3,1}\delta_{2,1}\delta_{13,12}}{\delta_{13,0}\delta_{12,0}\delta_{123,1}} d_{123,12,1,0}. \tag{11}$$

Later we will need the following technical

Lemma 1. *For the spherical cube we have the following identities*

$$1 = \frac{\delta_{123,2}\,\delta_{13,12}\,\delta_{3,1}\,\delta_{23,0}}{\delta_{1,123}\,\delta_{0,13}\,\delta_{2,3}\,\delta_{12,23}}, \tag{12}$$
$$1 = \frac{\delta_{2,1}\,\delta_{12,13}\,\delta_{123,3}\,\delta_{23,0}}{\delta_{0,12}\,\delta_{1,123}\ ,\delta_{13,23}\,\delta_{3,2}}, \tag{13}$$
$$1 = \frac{\delta_{123,2}\,\delta_{13,23}\,\delta_{1,3}\,\delta_{12,0}}{\delta_{23,12}\,\delta_{3,123}\,\delta_{0,13}\,\delta_{2,1}}. \tag{14}$$

Moreover, we have the following equivalent definition for the weight w_{123}

$$w_{123} = \frac{\delta_{1,3}\,\delta_{2,3}\,\delta_{13,23}}{\delta_{0,13}\,\delta_{0,23}\,\delta_{3,123}} d_{123,13,3,0}, \tag{15}$$
$$w_{123} = \frac{\delta_{1,2}\,\delta_{2,3}\,\delta_{12,23}}{\delta_{0,12}\,\delta_{0,23}\,\delta_{2,123}} d_{123,23,2,0}. \tag{16}$$

Proof. For the proof of the first identity we can apply inversion with the center in the point a_0 to the spherical cube. We denote by a'_i corresponding transformed points. We note that after inversion the sphere of the cube become a plane, points $\{a'_1, a'_{12}, a'_2\}$, $\{a'_2, a'_{23}, a'_3\}$, $\{a'_3, a'_{13}, a'_1\}$ are on a triangle with vertices $\{a'_1, a'_2, a'_3\}$ and three circles c'_i in the plane intersect in the point a'_{123}, where c'_i is the circle through three points a'_i, a'_{ij}, a'_{ik}.

Using the sinus theorem for a circular quadrangular $\{\{a'_1, a'_{13}, a'_{123}, a'_{12}\}\}$ we have

$$\frac{|a'_1 - a'_{123}|}{|a'_{12} - a'_{13}|} = \frac{\sin(\angle a'_1 a'_{12} a'_{123})}{\sin(\angle a'_{12} a'_1 a'_{13})}.$$

Similarly, we see that

$$\frac{|a'_{12} - a'_{23}|}{|a'_2 - a'_{123}|} = \frac{\sin(\angle a'_{12} a'_2 a'_{23})}{\sin(\angle a'_2 a'_{12} a'_{123})}, \quad \frac{|a'_2 - a'_3|}{|a'_1 - a'_3|} = \frac{\sin(\angle a'_1 a'_2 a'_3)}{\sin(\angle a'_2 a'_1 a'_3)}.$$

Since we have the following identities for angles

$$\angle a'_1 a'_2 a'_3 = \angle a'_{12} a'_2 a'_{23}, \qquad \angle a'_2 a'_1 a'_3 = \angle a'_{12} a'_1 a'_{13}, \qquad \angle a'_2 a'_{12} a'_{123} = \pi - \angle a'_1 a'_{12} a'_{123}.$$

Multiplication of right sides gives

$$\frac{|a'_1 - a'_{123}|}{|a'_{12} - a'_{13}|} \frac{|a'_{12} - a'_{23}|}{|a'_2 - a'_{123}|} \frac{|a'_2 - a'_3|}{|a'_1 - a'_3|} = 1. \tag{17}$$

We can apply a *distance formula* for inversion: $|A'B'| = r^2|AB|/(|OA||OB|)$, where O is a center and r is a radius of inversion; A', B' are points obtained after inversion of the points A, B. From the equality (17) using the distance formula we get

$$\frac{|a_1 - a_{123}|}{|a_{12} - a_{13}|} \frac{|a_{12} - a_{23}|}{|a_2 - a_{123}|} \frac{|a_2 - a_3|}{|a_1 - a_3|} \frac{|a_0 - a_{13}|}{|a_0 - a_{23}|} = 1. \tag{18}$$

This is the first identity in Lemma 1. Analogous, we have identities similar to the identity (17) and using the distance formula we get identities (13), (14).

Now we explain how to prove that the weight w_{123} defined by the formula (11) is equal to w_{123} defined by the another formula (15). The proof of the formula (16) is similar. First of all using the property (E) for the normalized difference (see [4, Section 4]) (i.e. $d_{123,12,1} = -d_{123,13,1}$ and so on) we have

$$d_{123,12,1,0} = -d_{123,13,1,0} = d_{123,13,3,0}. \tag{19}$$

If we compare two real coefficient in the formula (11) and (15) we see that they are equal because of the identity (13).

For the simplicity of notation in tri-linear map we will use the following indexing for points and weights

$$000 = 0, \qquad 001 = 1, \qquad 010 = 2, \qquad 100 = 3, \qquad 011 = 12,$$
$$101 = 13, \qquad 110 = 23, \qquad 111 = 123$$

i.e. $a_{000} = a_0$, $w_{000} = w_0$, $a_{001} = a_1$, $w_{001} = w_1$ and so on. Let we define the tri-linear map

$$f(s,t,u) = \left(\sum_{i,j,k=0}^{1} a_{ijk} w_{ijk} s_i t_j u_k\right)\left(\sum_{i,j,k=0}^{1} w_{ijk} s_i t_j u_k\right)^{-1}, \quad \text{where}$$
$$s_0 = s, \quad s_1 = 1-s, \quad t_0 = t, \quad t_1 = 1-t, \quad u_0 = u, \quad u_1 = 1-u.$$

We have the following main theorem

Theorem 1. *For the tri-linear map $f(s,t,u)$ we have*

$$f(1,s,t) = D_{H_3}(s,t), \qquad f(s,1,t) = D_{H_2}(s,t), \qquad f(s,t,1) = D_{H_1}(s,t), \tag{20}$$
$$f(0,s,t) = D_{H_3^*}(s,t), \qquad f(s,0,t) = D_{H_2^8}(s,t), \qquad f(s,t,0) = D_{H_1^*}(s,t), \tag{21}$$

i.e. tri-linear map is real on every side of spherical cube. Here

$$H_3 = \{(a_{0ij}, w_{0ij}),\ i,j = 0,1\}, \qquad H_2 = \{(a_{i0j}, w_{i0j}),\ i,j = 0,1\},$$
$$H_1 = \{(a_{ij0}, w_{ij0}),\ i,j = 0,1\}, \qquad H_3^* = \{(a_{1ij}, w_{1ij}),\ i,j = 0,1\},$$
$$H_2^* = \{(a_{i1j}, w_{i1j}),\ i,j = 0,1\}, \qquad H_1^* = \{(a_{ij1}, w_{ij1}),\ i,j = 0,1\}.$$

Proof. The first three equalities (20) follows by the definition of tri-linear map. Let us show that $f(s,t,0) = D_{H_1^*}(s,t)$.

We write the weights explicitly

$$w_i = d_{0,i} v_{0i} \quad i = 1,2,3, \qquad w_{12} = \frac{\delta_{1,2}}{\delta_{12,0}} d_{12,1,0} v_{03},$$
$$w_{13} = \frac{\delta_{1,3}}{\delta_{13,0}} d_{13,3,0} v_{02}, \qquad w_{23} = \frac{\delta_{2,3}}{\delta_{23,0}} d_{23,2,0} v_{01}.$$

Now we can find the weights $\{u_1, u_{12}, u_{123}, u_{13}\}$ for the patch defined the triple $T = \{\{a_1, a_{12}, a_{123}, a_{13}\}, v_{12}, v_{13}\}$, where $v_{ij} = d_{i,j} v_{0j} d_{ij}$, $i,j = 1,2,3$ is reflection of the vector v_{0j} along a line segment $\{a_i, a_j\}$. By the formulas (9) we have

$$u_1 = 1, \qquad u_{12} = d_{1,12}\, v_{12}, \qquad u_{123} = \frac{\delta_{12,13}}{\delta_{123,1}} d_{123,13,1}\, v_{13}\, v_{12}, \qquad u_{13} = d_{1,13}\, v_{13}.$$

Now we change the weights $\{u_1, u_{12}, u_{123}, u_{13}\}$ to weights

$$\left\{u_1 w_1, \frac{\delta_{1,2}}{\delta_{12,0}} u_{12} w_1, \frac{\delta_{1,2}\delta_{1,3}}{\delta_{12,0}\delta_{13,0}} u_{123} w_1, \frac{\delta_{1,3}}{\delta_{13,0}} u_{13} w_1\right\} = \{w_1, w_{12}, w_{123}, w_{13}\}.$$

The last equality can be checked by elementary computations using properties of normalized difference $d_{i,j}$ (see [4, Section 4]). As we already observed this change of weights does not change the patch, i.e $D_{H_1^*} = f(s,t,0)$. For the proof of the rest two formulas we can use the equivalent formulas (15), (16) for the weight w_{123}.

Using the MAPLE we checked that $f(s,t,u)$ is in $\mathbb{R}^3$ for many data. Therefore we conjecture that $f(s,t,u)$ is in $\mathbb{R}^3$ always (see Fig. 1).

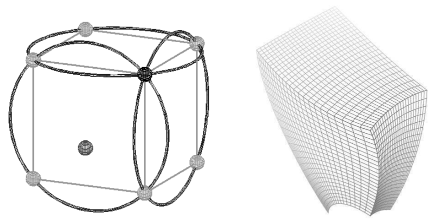

Fig. 1. A spherical and a cyclidic cube.

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REZIUMĖ

Kvaternioninės Bézier kreivės, paviršiai ir tūriai

Severinas Zubė

Darbe yra nagrinėjamas kvaternioninis tritiesinis Bézier tūris, kuris konstruojamas naudojant sferino kubo kontrolinius taškus. Darbe gautos formulės atitinkamų kontrolinių taškų svoriams rasti. Kadangi taikymuose dažniausiai yra reikalingi objektai trimatėje erdvėje, todėl nagrinėjamas klausimas: kada tritiesinis atvaizdis yra trimatėje erdvėje?

Raktiniai žodžiai: Bézier parametrizacija, kvaternionai.