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A fundamental equation

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Abstract. Very important and instructive trigonometric equation is considered. The solutions from different manuals and reference books are examined, mistakes of these solutions are discovered. It is shown how to make the solution of the equation shorter, how to present the answer more visually.

Keywords: trigonometric equations, inverse trigonometric functions, different methods of solution, answer forms.

At present, in the school is very little attention given to trigonometric equations, so for the reader it will be quite useful to get acquainted with one of the classic equations

$$\mathrm{tg}(\pi \mathrm{tg} x) = \mathrm{ctg}(\pi \mathrm{ctg} x). \tag{1}$$

The equation has many merits. One – it is very aesthetically pleasing, and that could compete with it unless its twin – equation

$$\sin(\pi \sin x) = \cos(\pi \cos x),$$

while the latter just much simpler. Second – solving it requires substantial knowledge of trigonometry – inverse trigonometric functions, these connecting relationships, definition domains, value ranges and so on. Third, it faced dealing with serious logical difficulties related to the issues of equations equivalence.

An interesting history of this equation. Probably the first time it was given during the entrance exams to the Moscow Lomonosov University, then got into the well reputable directories and tasks encyclopedic publications. Unfortunately, we pierced them convinced that by equation solving and interpreting has been made everywhere vexatious mistakes. The least quantity of these is made in the books [1] and [2]. By the way, the solution is actually everywhere copyed from [1] down, and the mistakes in enacting solution only confirms the level of difficulty of the equation.

We provide a solution of [1], only indicate how it can be improved, how more aesthetic to record the answers. It is also very important for developing mathematical taste – the latter question is unnecessary in most cases ignored. But the main thing – we were able to find a new, shorter solution and get a simpler answer – as mentioned, in the literature is exposed the same solution and is indicated the same answer from [1]. Incidentally, different forms of the answer is often by solving trigonometric equations in several ways.

Begin to solve equation (1) absolutely necessary from the definition domain defined by the relationships

$$\mathrm{tg}x \neq m + \frac{1}{2}, \quad \mathrm{ctg}x \neq p \quad (m, p \in \mathbb{Z}). \tag{2}$$

There equation (1) is equivalent to the equations $\mathrm{tg}(\pi\mathrm{tg}x) = \mathrm{tg}(\frac{\pi}{2} - \pi\mathrm{ctg}x)$, $\pi\,\mathrm{tg}\,x = \frac{\pi}{2} - \pi\,\mathrm{ctg}\,x + k\pi$,

$$\mathrm{tg}x = \frac{1}{2} - \frac{1}{\mathrm{tg}x} + k, \tag{3}$$

$$2\mathrm{tg}^2x - (2k+1)\mathrm{tg}x + 2 = 0,$$

$$\mathrm{tg}x = \frac{1}{4}\big(2k + 1 \pm \sqrt{(2k+1)^2 - 16}\,\big), \quad k \in \mathbb{Z}. \tag{4}$$

Thus, solving equation (1) we need to reject $\mathrm{tg}x$ values not covered by the definition domain (2), i.e. those involving $\mathrm{tg}x$ of the form $m + \frac{1}{2}$ or $\frac{1}{m}$. (By the way, no way here instead of m writing k like that done in the book [1] – m and k have nothing in common.)

Let us define set of k values where $\mathrm{tg}\,x$ is rational (then it will be easier to figure out when they are of the form $m + \frac{1}{2}$ or $\frac{1}{m}$). First of all, it should be $(2k+1)^2 \geqslant 16$, thus values $-2, -1, 0, 1$ must be rejected. Second, $\mathrm{tg}\,x$ will be rational only, when $(2k+1)^2 - 16$ will be the square of integer. Since this difference is odd, it should be

$$(2k+1)^2 - 16 = (2a+1)^2, \quad \text{or} \quad (2k+1)^2 - (2a+1)^2 = 16,$$

$$(k-a)(k+a+1) = 4.$$

It is now possible to solve number of systems $k - a = \ldots, k + a + 1 = \ldots$ (this suggests the authors in the books [1, 2, 3]), but you can immediately determine the appropriate values of k and significantly shorten the solution. The figures $(2k+1)^2$ ir $(2a+1)^2$ depends to the sequence of odd squares $1, 9, 25, 49, 81, \ldots$. Differences between adjacent members increases and beginning from 49 are larger than 16. Even greater are differences between non-adjacent members of the sequence. Hence, the difference 16 provides only $5^2 - 3^2$, so $(2k+1)^2 = 5^2$, $2k+1 = \pm 5$, $k = -3$ and $k = 2$.

Thus, the rational tangent in formula (4) provides only $k = -3$ and $k = 2$. We have to determine when the k values give tangents of forms $\frac{2m+1}{2}$ or $\frac{1}{m}$.

When $k = -3$, equation (4) turns to $\mathrm{tg}x = \frac{1}{4}(-5 \pm 3)$, i.e. $\mathrm{tg}\,x = -2$ and $\mathrm{tg}\,x = -\frac{1}{2}$. The value -2 is impossible to write neither as $\frac{2m+1}{2}$, nor as $\frac{1}{m}$, where m is integer (in fact, the equations $\frac{2m+1}{2} = -2$ and $\frac{1}{m} = -2$ have no integer solutions). Hence, $\mathrm{tg}\,x = -2$ is suited to the our equation (1) and gives solutions $x = n\pi - \mathrm{arctg}\,2$. The value $-\frac{1}{2}$ can be written as $\frac{1}{-2}$, thus $\mathrm{tg}\,x = -\frac{1}{2}$ does not suit to equation (1).

When $k = 2$, equation (4) turns to $\mathrm{tg}\,x = \frac{1}{4}(5 \pm 3)$, i.e. $\mathrm{tg}\,x = 2$ and $\mathrm{tg}\,x = \frac{1}{2}$. The value 2 is neither of form $\frac{2m+1}{2}$, nor of form $\frac{1}{m}$, thus $\mathrm{tg}\,x = 2$ is suitable to equation (1), $x = n\pi + \mathrm{arctg}\,2$. The value $\frac{1}{2}$ is of form $\frac{1}{m}$, so it has to be rejected.

Thus, we will definitely fit tangent values of both signs "+" and "−" for all integer k with the exception of $-3, -2, -1, 0, 1, 2$. With $k = -3$ and $k = 2$ suites only one of signes, thus they must be included to answer separately. The answer can be written as

$$x \in \left\{ n\pi \pm \operatorname{arctg}2,\ n\pi + \operatorname{arctg}\frac{2k+1 \pm \sqrt{(2k+1)^2 - 16}}{4} \right\}, \tag{5}$$
$$n \in \mathbb{Z},\ k = 3, \pm 4, \pm 5, \ldots.$$

So it is written in the books [1] and [2], only in the book [1] there is a proofreading error – not rejected value $k = 1$. There are a lot of proofreading errors (and not only proofreading) in the book [3], and before the second arctangent the sign $\pm$ is replaced, thus in so written answer some values of x are included twice. For example, when $k = 3$, we have $x = n\pi \pm \operatorname{arctg} \frac{7\pm\sqrt{33}}{4}$, and when $k = -4$, we have $x = n\pi \pm \operatorname{arctg} \frac{-7\pm\sqrt{33}}{4}$, – after all, it's the same!

In the beginning we promised to write an answer nicer. Here k values are scattered, begin from 3. Write sequence $(2k+1)$ with the values $k = 3, 4, 5, \ldots.$ and the sequence $(2k+1)$ with the values $k = -4, -5, -6, \ldots$:

$$7, 9, 11, \ldots,$$
$$-7, -9, -11, \ldots.$$

It can be seen that they can be written as a sequence $(2k+5)$ with values $k = 1, 2, 3, \ldots$ and as the sequence – $(2k+5)$ with values $k = 1, 2, 3 \ldots.$ Connecting can be a single formula $\pm(2k+5)$, $k \in \mathbb{N}$. Now our answer would look like this:

$$\left\{ n\pi \pm \operatorname{arctg}2,\ n\pi \pm \operatorname{arctg}\frac{2k+5 \pm \sqrt{(2k+5)^2 - 16}}{4},\ n \in \mathbb{Z},\ k \in \mathbb{N} \right\}, \tag{6}$$

or that there be no doubts about the combinations of signs (we had in mind all 4 variants), such as:

$$\left\{ n\pi \pm \operatorname{arctg}2,\ n\pi \pm \operatorname{arctg}\frac{2k+5 - \sqrt{(2k+5)^2 - 16}}{4}, \right.$$
$$\left. n\pi \pm \operatorname{arctg}\frac{2k+5 + \sqrt{(2k+5)^2 - 16}}{4},\ n \in \mathbb{Z},\ k \in \mathbb{N} \right\}. \tag{7}$$

In this way, notice that the first series of solutions can be obtained from the third by taking $k = 0$. Thus the two can be combined into one:

$$\left\{ n\pi \pm \operatorname{arctg}\frac{2k+5 - \sqrt{(2k+5)^2 - 16}}{4},\ n\pi \pm \operatorname{arctg}\frac{2m+5 + \sqrt{(2m+5)^2 - 16}}{4}, \right.$$
$$\left. k \in \mathbb{N},\ m \in \mathbb{N} \cup \{0\},\ n \in \mathbb{Z} \right\}. \tag{8}$$

By the way, the reader can find that a form of the answer as follows is more representative:

$$\left\{ n\pi \pm \operatorname{arctg}\frac{2k+5 - \sqrt{(2k+5)^2 - 16}}{4},\ n\pi \pm \operatorname{arctg}\frac{2k+3 + \sqrt{(2k+3)^2 - 16}}{4}, \right.$$
$$\left. n \in \mathbb{Z},\ k \in \mathbb{N} \right\}. \tag{9}$$

It turns out that there is a shorter solution of equation (1), and answer structure is simpler. Return to equation (3). The solutions we need to reject those when $\operatorname{tg} x = m + \frac{1}{2}$, and those when $\operatorname{ctg} x = p$ $(m, p \in \mathbb{Z})$. However, if equation (3) has a solution $\operatorname{ctg} x = p$, it is then $\operatorname{tg} x + p = k + \frac{1}{2}$, $\operatorname{tg} x = k - p + \frac{1}{2}$. Thus excluding $\operatorname{tg} x = m + \frac{1}{2}$ from solutions, we exclude $\operatorname{ctg} x = p$ too. Thus, from solutions of equation (3) is sufficient to exclude the solutions to which $\operatorname{tg} x = m + \frac{1}{2}$.

Further, if $\operatorname{tg} x = m + \frac{1}{2}$, then $\operatorname{ctg} x = \frac{2}{2m+1}$, and from equation (3) $m + \frac{1}{2} + \frac{2}{2m+1} = k + \frac{1}{2}$,

$$\frac{2}{2m+1} = k - m. \tag{10}$$

It means that $\frac{2}{2m+1}$ is integer, which is possible only when $2m+1$ is divisor of 2, i.e., is equal to one of the numbers $1, -1, 2, -2$. But $2m+1$ – odd number, thus $2m+1 = 1$ or $2m+1 = -1$, i.e. $m = 0$ or $m = -1$. When $m = 0$, then from equality (10) $k = 2$, and when $m = -1$, then $k = -3$.

So even without starting to solve equation (3), we have got an interesting result: if $k = 2$, then for equation (1) extraneous solutions may be only solutions of equation (3) of form $\operatorname{tg} x = m + \frac{1}{2} = 0 + \frac{1}{2} = \frac{1}{2}$ (if there are any); if $k = 3$, then for equation (1) extraneous solutions can be only solutions of equation (3) of form $\operatorname{tg} x = m + \frac{1}{2} = -1 + \frac{1}{2} = -\frac{1}{2}$ (if any); for other k values we have no extraneous solutions. This, incidentally, was established above by analysis of formula (4).

Now solve equation (3) otherwise:

$$\frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} = k + \frac{1}{2}, \qquad \frac{1}{\sin x \cos x} = \frac{2k+1}{2},$$
$$\sin x \cos x = \frac{2}{2k+1}, \qquad \sin 2x = \frac{4}{2k+1}.$$

The latter equation has no solutions when $|\frac{4}{2k+1}| > 1$, i.e., when $|2k+1| < 4$, $-4 < 2k+1 < 4$, $-5 < 2k < 3$, $-\frac{5}{2} < k < \frac{3}{2}$, $k \in \{-2 -1, 0, 1\}$. For other values of k we get:

$$2x = 2n\pi + \arcsin\frac{4}{2k+1} \quad \text{and} \quad 2x = (2n+1)\pi - \arcsin\frac{4}{2k+1},$$

i.e.

$$x = n\pi + \frac{1}{2}\arcsin\frac{4}{2k+1} \quad \text{and} \quad x = \frac{(2n+1)}{2}\pi - \frac{1}{2}\arcsin\frac{4}{2k+1}.$$

Now we have to reject the extraneous solutions. As we have already set up, they can be just as $k = 2$, $\operatorname{tg} x = \frac{1}{2}$ and $k = -3$, $\operatorname{tg} x = -\frac{1}{2}$.

When $k = 2$, we have solutions

$$x = n\pi + \frac{1}{2}\arcsin\frac{4}{5} \quad \text{and} \quad x = n\pi + \frac{\pi}{2} - \frac{1}{2}\arcsin\frac{4}{5}. \tag{11}$$

Determine, when $\operatorname{tg} x = \frac{1}{2}$. Denote $\arcsin\frac{4}{5} = \alpha$. Because $0 < \alpha < \frac{\pi}{2}$ and $\sin\alpha = \frac{4}{5}$, then

$$\cos\alpha = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}, \qquad \operatorname{tg}\frac{\alpha}{2} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{1}{2}, \qquad \operatorname{tg}\left(\frac{\pi}{2} - \frac{\alpha}{2}\right) = \operatorname{ctg}\frac{\alpha}{2} = 2.$$

Thus the first series of (11) must be rejected, and remains the second series $x = \frac{(2n+1)\pi}{2} - \arcsin\frac{4}{5}$.

When $k=-3$, the solutions are

$$x = n\pi - \frac{1}{2}\arcsin\frac{4}{5} \quad \text{and} \quad x = n\pi + \frac{\pi}{2} + \frac{1}{2}\arcsin\frac{4}{5}. \tag{12}$$

Now for the first series $\operatorname{tg} x = -\frac{1}{2}$, and for a second series $\operatorname{tg} x = \operatorname{tg}(\frac{\pi}{2}+\alpha) = -\operatorname{ctg}\frac{\alpha}{2} = -2$. Thus, the second series in (12) has to be rejected, and remains the first series $x = (2n)\frac{\pi}{2} - \frac{1}{2}\arcsin\frac{4}{5}$.

Join solutions of cases $k=2$ and $k=-3$. Since in one series $(2n+1)$ are odd numbers, in the other $(2n)$ are even, they can be combined into a single series:

$$x = \frac{n\pi}{2} - \frac{1}{2}\arcsin\frac{4}{5}.$$

Now we can write the final answer: a set of solutions of equation (1) is

$$\left\{\frac{n\pi}{2} - \frac{1}{2}\arcsin\frac{4}{5},\ \frac{n\pi}{2} + \frac{(-1)^n}{2}\arcsin\frac{4}{2k+1},\ k = 3, \pm4, \pm5, \ldots, n\in\mathbb{Z}\right\}. \tag{13}$$

We have already seen how you can make sure that it is the same (5) set. Here, the structure of the answer is much simpler – no need of any square root (by the way, above in solving equation was actually used obvious, but rarely formulated and substantiated theorem: the square root of a natural number is rational only when this number is the square of an integer). Of course, this refers to the fact that equation (3) is quadratic with respect to the $\operatorname{tg} x$ but with respect to $\sin 2x$ – first degree.

Just as the (5) can be written as (6) – (9), set (13) can be represented by different variants. Then, for example, formula (6) corresponds to a variant as follows:

$$\left\{\frac{n\pi}{2} - \frac{1}{2}\arcsin\frac{4}{5},\ \frac{n\pi}{2} \pm \frac{1}{2}\arcsin\frac{4}{2k+5},\ n\in\mathbb{Z},\ k\in\mathbb{N}\right\}.$$

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REZIUMĖ

Fundamentali lygtis

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Supažindinama su svarbia ir pamokoma trigonometrine lygtimi. Apžvelgti mokymo priemonėse pateikti sprendimai, nurodytos tų sprendimų klaidos. Parodyta, kaip galima sprendimą sutrumpinti, vaizdžiau užrašyti atsakymą.

Raktiniai žodžiai: trigonometrinės lygtys, atvirkštinės trigonometrinės funkcijos, sprendimo metodų įvairovė, atsakymo pavidalai.