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The discount version of large deviations for a randomly indexed sum of random variables

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Abstract. In this paper, we consider a compound random variable $Z = \sum_{j=1}^{N} v^j X_j$, where $0 < v < 1$, $Z = 0$, if $N = 0$. It is assumed that independent identically distributed random variables $X_1, X_2, \dots$ with mean $\mathbf{E}X = \mu$ and variance $\mathbf{D}X = \sigma^2 > 0$ are independent of a non-negative integer-valued random variable N. It should be noted that, in this scheme of summation, we must consider two cases: $\mu \neq 0$ and $\mu = 0$. The paper is designated to the research of the upper estimates of normal approximation to the sum $\tilde{Z} = (Z - \mathbf{E}Z)(\mathbf{D}Z)^{-1/2}$, theorems on large deviations in the Cramer and power Linnik zones and exponential inequalities for $\mathbf{P}(\tilde{Z} \geqslant x)$.

Keywords: cumulant, large deviations theorems, discounted limit theorems, normal approximation, random number of summands.

Main conditions and results

Let's say that N is a non-negative integer-valued random variable (r.v.) with mean $\mathbf{E}N = \alpha$, variance $\mathbf{D}N = \beta^2$, and the distribution $\mathbf{P}(N = l) = p_l$, $l = 0, 1, \dots$. In addition, $\{X, X_j,\ j = 1, 2, \dots\}$ is a family of independent identically distributed (i.i.d.) random variables (r.v.'s) with mean $\mathbf{E}X = \mu$, variance $\mathbf{D}X = \sigma^2 > 0$ and the distribution function $F(x) = \mathbf{P}(X < x)$, $x \in \mathbb{R}$. We assume that X_j are independent of N. Furthermore, the k-th order cumulants and the characteristic function of X will be denoted by $\Gamma_k(X)$, $k = 1, 2, \dots$, $\varphi_X(u) = \mathbf{E}\exp\{iuX\}$, $u \in \mathbb{R}$, respectively.

Denote a compound r.v. Z as follows

$$Z = \sum_{j=1}^{N} v^j X_j, \quad 0 < v < 1, \qquad Z = 0, \quad \text{if } N = 0.$$

The research of randomly weighted sums plays an important role in insurance, economic theory, finance mathematics and is essential in other fields too. For instance, in the ruin theory the weights are interpreted as discount factors and the sequence X_j as the net returns of an insurance company to analyse the probability of ruin either in a finite or infinite time (see, for example, [2]).

Denote,

$$T = \sum_{j=1}^{N} v^j, \qquad \tilde{T} = \sum_{j=1}^{N} v^{2j}, \tag{1}$$

$$\mathbf{E}v^{kN} = \sum_{l=0}^{\infty} v^{kl} p_l, \qquad \mathbf{D}v^{kN} = \mathbf{E}v^{2kN} - \left(\mathbf{E}v^{kN}\right)^2, \quad k = 1, 2, \ldots. \tag{2}$$

Note that $T = (1 - v^N)v/(1 - v)$, $\tilde{T} = (1 - v^{2N})v^2/(1 - v^2)$. Therefore, in view of (1), (2), we calculate

$$\mathbf{E}T = v\left(1 - \mathbf{E}v^N\right)/(1 - v), \qquad \mathbf{E}\tilde{T} = v^2\left(1 - \mathbf{E}v^{2N}\right)/\left(1 - v^2\right),$$
$$\mathbf{E}T^2 = v^2\left(\mathbf{E}v^{2N} - 2\mathbf{E}v^N + 1\right)/(1 - v)^2, \qquad \mathbf{D}T = v^2\mathbf{D}v^N/(1 - v)^2.$$

According to (5), (6) in [1], we get

$$\mathbf{E}Z = \mu\mathbf{E}T, \qquad \mathbf{D}Z = \sigma^2\mathbf{E}\tilde{T} + \mu^2\mathbf{D}T.$$

We say that the r.v. X with $\sigma^2 > 0$ satisfies condition (B_γ) if there exist constants $\gamma \geqslant 0$ and $K > 0$ such that

$$\left|\mathbf{E}X^k\right| \leqslant (k!)^{1+\gamma} K^{k-2}\mathbf{E}X^2, \quad k = 3, 4, \ldots. \tag{B_γ}$$

Note that $\mathbf{E}(X - \mu) = 0$, $\mathbf{E}(X - \mu)^2 = \sigma^2$, therefore (B_γ) implies $|\mathbf{E}(X - \mu)^k| \leqslant (k!)^{1+\gamma}K^{k-2}\sigma^2$. Taking into consideration that $\Gamma_k(X) = \Gamma_k(X - \mu)$, $k = 2, 3, \ldots$ and according to Lemma 3.1 in [6], we get

$$\left|\Gamma_k(X)\right| \leqslant (k!)^{1+\gamma} M^{k-2}\sigma^2, \quad k = 3, 4, \ldots, \tag{S}$$

where $M = 2(\sigma \vee K)$. Furthermore, we assume that the r.v. T satisfies condition (L) if there exist constants $K_1 > 0$ and $p \geqslant 0$ such that

$$\left|\Gamma_k(T)\right| \leqslant (1/2)k!K_1^{k-2}(\mathbf{D}T)^{1+(k-2)p}, \quad k = 2, 3, \ldots. \tag{L}$$

It should be noted that here condition (L) is used if $\mu \neq 0$. And if $\mu = 0$, then for the r.v. $\tilde{T}$ we use the following condition $(\tilde{\mathrm{L}})$

$$\left|\Gamma_k(\tilde{T})\right| \leqslant k!K_2^{k-1}(\mathbf{E}\tilde{T})^{1+(k-1)p}, \quad k = 1, 2, \ldots, \tag{$\tilde{\mathrm{L}}$}$$

where $K_2 > 0$.

It will be observed that, to achieve the purpose of this paper, we have to use the cumulant method (see [6]) developed by R. Rudzkis, L. Saulis, V. Statulevičius (1978). The cumulant method is good in the investigation of large deviations for randomly indexed sums of both independent and dependent r.v.'s. Since we are interested not only in the convergence to the normal distribution, but also in a more accurate asymptotic analysis of the distribution, we must find an accurate upper estimate for $\Gamma_k(\tilde{Z})$, $k = 3, 4, \ldots$, where $\tilde{Z} = (Z - \mathbf{E}Z)(\mathbf{D}Z)^{-1/2}$. And afterwards, we can use general lemmas on the behaviour of $F_{\tilde{Z}}(x) = \mathbf{P}(\tilde{Z} < x)$.

Recall that $0 < v < 1$ and denote $(b \vee c) = \max\{b, c\}b$, $c \in \mathbb{R}$.

Lemma 1. *If, for the r.v. X with mean $\mu \neq 0$ and variance $\sigma^2 > 0$, condition* (B_γ) *is fulfilled and the r.v. T satisfies condition* (L), *then*

$$\left|\Gamma_k(\tilde{Z})\right| \leqslant (k!)^{1+\gamma}/\Delta_v^{k-2}, \quad k = 3, 4, \ldots \tag{3}$$

where

$$\Delta_v = L_v^{-1}\sqrt{\mathbf{D}Z}, \qquad L_v = \left(2K_1|\mu|(\mathbf{D}T)^p \vee \left(1 \vee \sigma/\left(2|\mu|\right)\right)2vM\right). \tag{4}$$

If $\mu = 0$ and X, $\tilde{T}$ satisfy conditions (B_γ), $(\tilde{\mathrm{L}})$, *then estimate* (3) *holds with $\Delta_v = \tilde{\Delta}_v$, where*

$$\tilde{\Delta}_v = \tilde{L}_v^{-1}\sqrt{\mathbf{D}Z}, \qquad \tilde{L}_v = \left(2 \vee K_2(\mathbf{E}\tilde{T})^p\right)(1/2 \vee v)M. \tag{5}$$

In the current paper, we obtain the accuracy of approximation of the distribution function $F_{\tilde{Z}}(x)$ by the standard normal distribution function $\Phi(x)$.

Denote

$$\Delta_{v,\gamma} = c_\gamma \Delta_v^{1/(1+2\gamma)}, \qquad c_\gamma = (1/6)(\sqrt{2}/6)^{1/(1+2\gamma)}.$$

Theorem 1. *Let X with $\mu \neq 0$, $\sigma^2 > 0$, and T satisfy conditions* (B_γ) *and* (L), *respectively. Then*

$$\sup_x \left|F_{\tilde{Z}}(x) - \Phi(x)\right| \leqslant 18/\Delta_{v,\gamma}. \tag{6}$$

Moreover, the large deviations theorem in the Cramer and power Linnik zones is proved and exponential inequalities are obtained.

Theorem 2. *Let X with $\mu \neq 0$, $\sigma^2 > 0$, and T satisfy conditions* (B_γ) *and* (L), *respectively. Then the relations*

$$\frac{1 - F_{\tilde{Z}}(x)}{1 - \Phi(x)} \to 1, \qquad \frac{F_{\tilde{Z}}(-x)}{\Phi(-x)} \to 1, \tag{7}$$

hold for $x \geqslant 0$, $x = o((\mathbf{D}T)^{(1/2-p)\nu(\gamma)})$, as $v \to 1$, and $\beta^2 \to \infty$. Here $\nu(\gamma) = (1+2(1\vee\gamma))^{-1}$, $0 \leqslant p < 1/2$. If $\gamma = 0$, then (7) *hold for $x \geqslant 0$, $x = o((\mathbf{D}T)^{(1/2-p)/3})$.*

Theorem 3. *Let X with $\mu \neq 0$, $\sigma^2 > 0$, and T satisfy conditions* (B_γ) *and* (L), *respectively. Then the exponential inequalities*

$$\mathbf{P}(\pm\tilde{Z} \geqslant x) \leqslant \begin{cases} \exp\{-x^2/2^{3+\gamma}\}, & 0 \leqslant x \leqslant (2^{(1+\gamma)^2}\Delta_v)^{1/(1+2\gamma)}, \\ \exp\{-(x\Delta_v)^{1/(1+\gamma)}/4\}, & x \geqslant (2^{(1+\gamma)^2}\Delta_v)^{1/(1+2\gamma)} \end{cases} \tag{8}$$

are valid.

Remark 1. If $\mu = 0$ and for the r.v.'s X, $\tilde{T}$ conditions (B_γ), $(\tilde{\mathrm{L}})$ are fulfilled, then estimates (6), (8) hold with $\Delta_v = \tilde{\Delta}_v$. Furthermore, (7) hold for $x \geqslant 0$, $x = o((\mathbf{E}\tilde{T})^{(1/2-p)\nu(\gamma)})$, as $v \to 1$, and $\alpha \to \infty$.

Remark 2. If $N = N_t$, $t \geqslant 0$ is a homogeneous Poisson process with intensity $\lambda > 0$, then $\mathbf{E}\tilde{T}$, $\mathbf{D}T$ hold with $\mathbf{E}v^{2N_t} = e^{-\lambda t(1-v^2)}$, $\mathbf{D}v^{N_t} = e^{-\lambda t(1-v^2)}(1 - e^{-\lambda t(1-v)^2})$. Accordingly, (7) hold if $v \to 1$, and $t \to \infty$.

If N is distributed according to the binomial law with the parameters $0 < p_1 < 1$, $n = 0, 1, \ldots$, then $\mathbf{E}\tilde{T}$, $\mathbf{D}T$ hold with $\mathbf{E}v^{2N} = (1 - p_1(1 - v^2))^n$, $\mathbf{D}v^N = (1 - p_1(1 - v^2))^n - (1 - p_1(1 - v))^{2n}$. Therefore, (7) hold if $v \to 1$, and $n \to \infty$.

Note that, for the r.v. Z in the case $\mu \neq 0$, $v^j \equiv a_j$, $j = 1, 2, \ldots, 0 \leqslant a_j < \infty$, and in the case $a_j \equiv 1$, large deviations theorems in the Cramer zone have been proved in the papers [1, 5]. Besides, large deviation theorems in a discounted version when N is a non-random variable have been proved in [4].

Proofs of Lemma 1 and Theorems 1–3

Proof of Lemma 1. Since i.i.d. r.v.'s X_j, $j = 1, 2, \ldots$ and N are independent, the proof of Lemma 1 in [1], yields

$$\varphi_Z(u) = \mathbf{E}e^{iuZ} = \sum_{l=0}^{\infty} \exp\left\{ \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{j=1}^{l} v^{jk} \Gamma_k(X)(iu)^k \right\} p_l, \quad u \in \mathbb{R}, \tag{9}$$

where $\sum_{j=1}^{l} v^{jk} = v^k(1 - v^{kl})/(1 - v^k)$. Based on Lemma 1 and equality (1.6) in [3], in addition, taking into account (9), we find that, for all $k = 1, 2, \ldots$,

$$\begin{aligned} \Gamma_k(Z) =& k! \sum\nolimits_1^* (-1)^{m-1}(m-1)! \\ & \times \prod_{s=1}^{k} \frac{1}{m_s!} \left(\sum_{l=1}^{\infty} \sum\nolimits_2^* \prod_{r=1}^{s} \frac{1}{\eta_r!} \left(\frac{1}{r!} \sum_{j=1}^{l} v^{jr} \Gamma_r(X) \right)^{\eta_r} p_l \right)^{m_s}, \end{aligned}$$

where $\sum_1^*$, $\sum_2^*$ denote a summation over all the non-negative integer solutions $m_1 + 2m_2 + \cdots + km_k = k$, $m = m_1 + \cdots + m_k$, and $\eta_1 + 2\eta_2 + \cdots + s\eta_s = s$, respectively. The application of $\sum_{j=1}^{l} v^{jr} \leqslant v^{(r-2)} \sum_{j=1}^{l} v^{2j}$, $|\Gamma_m(\tilde{T})| \leqslant v^m |\Gamma_m(T)|$ and conditions (S), (L), allows us to assert that

$$\big|\Gamma_k(Z)\big| \leqslant k! \sum\nolimits_1^* \big|\Gamma_m(\tilde{T})\big| \prod_{s=1}^{k} \frac{1}{m_s!} \left(\frac{1}{s!} v^{s-2} \big|\Gamma_s(X)\big| \right)^{m_s} \tag{10}$$

$$\begin{aligned} &\leqslant (k!)^{1+\gamma} (vM)^{k-2} \sigma^2 \mathbf{E}\tilde{T} + \frac{1}{2} k! \mathbf{D}T \sum\nolimits_4^* \frac{\tilde{m}!}{m_1! \cdot \cdots \cdot m_{k-1}!} \\ &\cdot \times \big(K_1(\mathbf{D}T)^p\big)^{\tilde{m}-2} v^{\tilde{m}} \left(\frac{|\mu|}{v} \right)^{m_1} \prod_{s=2}^{k-1} \big((s!)^{\gamma} (vM)^{s-2} \sigma^2\big)^{m_s}. \end{aligned} \tag{11}$$

where $\Gamma_m(\tilde{T}) = m! \sum_3^* (-1)^{\tau-1}(\tau - 1)! \prod_{n=1}^{m} (1/\tau_n!)((1/n!)\mathbf{E}\tilde{T}^n)^{\tau_n}$, $\mathbf{E}\tilde{T}^n = \sum_{l=0}^{\infty} \big(\sum_{j=1}^{l} v^{2j} \big)^n p_l$. In addition, $\sum_3^*$ denotes a summation over all the non-negative integer solutions $\tau_1 + 2\tau_2 + \cdots + m\tau_m = m$, and $\tau = \tau_1 + \cdots + \tau_m$; $\sum_4^*$ is taken over all the non-negative integer solutions $m_1 + 2m_2 + \cdots + (k-1)m_{k-1} = k$, where $0 \leqslant m_1, \ldots, m_{k-1} \leqslant k$, and $\tilde{m} = m_1 + \cdots + m_{k-1}$. Here $k = 2, 3 \ldots, 2 \leqslant \tilde{m} \leqslant k$.

We need the equality $g_k = \sum_1^* m!/(m_1! \cdot \cdots \cdot m_k!) = 2^{k-1}$, $k = 1, 2 \ldots$ and inequality $a!b! \leqslant (a+b)!$. It is assumed by convention that $g_0 = 1$. Consequently, after evaluations when $\mu \neq 0$, we get

$$\sum\nolimits_4^* \frac{\tilde{m}!}{m_1! \cdot \cdots \cdot m_{k-1}!} = 2^{k-1} - 1, \qquad \prod_{s=1}^{k-1} (s!)^{m_s} \leqslant (k-1)!, \quad k = 2, 3, \ldots,$$

$$v^{\tilde{m}} \big(v^{-1}|\mu|\big)^{m_1} \prod_{s=2}^{k-1} \big((vM)^{s-2}\sigma^2\big)^{m_s} \leqslant |\mu|^{\tilde{m}} \Big(\big(1 \vee \sigma\big(2|\mu|\big)^{-1}\big) vM\Big)^{k-\tilde{m}}, \quad k = 3, 4, \ldots,$$

as $m_2+2m_3+\dots+(k-2)m_{k-1}=k-\tilde{m}$, $0\leqslant m_2+\dots+m_{k-1}\leqslant k-\tilde{m}$. By substituting these estimates into (11), we obtain $|\varGamma_k(Z)|\leqslant (k!)^{1+\gamma}L_v^{k-2}\mathbf{D}Z$, $k=3,4,\dots$. Finally, in accordance with $\varGamma_k(\tilde{Z})=(\mathbf{D}Z)^{-k/2}\varGamma_k(Z-\mathbf{E}Z)=(\mathbf{D}Z)^{-k/2}\varGamma_k(Z)$, we get estimate (3) with constants (4).

Now, suppose that $\mu=0$, and $0^0=1$. With reference to (10) and conditions (S), ($\tilde{\mathrm{L}}$), we find

$$\bigl|\varGamma_k(Z)\bigr|\leqslant k!\mathbf{E}\tilde{T}\sum\nolimits_5^*\frac{\bar{m}!}{m_2!\cdot\ldots\cdot m_k!}\bigl(K_2(\mathbf{E}\tilde{T})^p\bigr)^{\bar{m}-1}\prod_{s=2}^k\bigl((s!)^\gamma(vM)^{s-2}\sigma^2\bigr)^{m_s},$$

where the sum $\sum_5^*$ is taken over all the non-negative integer solutions $2m_2+\dots+km_k=k$, where $0\leqslant m_2,\dots,m_k\leqslant k$, and $\bar{m}=m_2+\dots+m_k$. Here $k=2,3,\dots$, $1\leqslant\bar{m}\leqslant k$. The next step is to evaluate similarly as in the case where $\mu\neq 0$. Afterwards, we derive

$$\sum\nolimits_5^*\frac{\bar{m}!}{m_2!\cdot\ldots\cdot m_k!}\leqslant 2^{k-2},\qquad \prod_{s=2}^k(s!)^{m_s}\leqslant k!,\quad k=2,3,\dots,$$
$$\prod_{s=2}^k\bigl((vM)^{s-2}\sigma^2\bigr)^{m_s}\leqslant\bigl(\sigma/(2v)\bigr)^{\bar{m}}(vM)^{k-\bar{m}},\quad k=2,3,\dots,$$

as $m_2+2m_3+\dots+(k-1)m_k=k-\bar{m}$. Accordingly, $|\varGamma_k(Z)|\leqslant(k!)^{1+\gamma}\tilde{L}_v^{k-2}\mathbf{D}Z$, $k=3,4,\dots$. In view of $\varGamma_k(\tilde{Z})=(\mathbf{D}Z)^{-k/2}\varGamma_k(Z)$, we prove Lemma 1. □

Proof of Theorem 1. The proof of Theorem 1 is obtained thanks to estimate (3). The compound r.v. $\tilde{Z}$ satisfies V. Statulevičius' condition (S_γ) (see [6]) with the parameter $\varDelta=\varDelta_v$. Thus Corollary 2.1 in [6] yields the proof of this theorem. □

Proof of Theorem 2. The statement of Theorem 2 follows immediately from Lemma 2.3 in [6] with $\varDelta=\varDelta_v$. We have to prove that $L_\gamma(x)\to 0$, $x/\varDelta_{v,\gamma}\to 0$, as $\varDelta_v\to\infty$, where $L_\gamma(x)$ is defined by (2.8) in [6].

Recalling the definitions of $\varDelta_v$, $\mathbf{E}\tilde{T}$, $\mathbf{D}T$, we get $\varDelta_v\geqslant C(\mathbf{D}T)^{1/2-p}$, $C>0$. Hence it follows that $0\leqslant p<1/2$, $\varDelta_v\to\infty$, in case $(\mathbf{D}T)^{1/2-p}\to\infty$. We have calculated that $\mathbf{D}T\to\infty$, as $v\to 1$ and $\beta^2\to\infty$. Thanks to estimate (3) and equality (2.9) in [6] for all $x=o((\mathbf{D}T_t)^{(1/2-p)\nu(\gamma)})$ with $\nu(\gamma)=(1+2(1\vee\gamma))^{-1}$, we obtain that $\lambda_3x^3=(1/6)\varGamma_3(\tilde{Z})x^3=o((\mathbf{D}T)^{2\nu(\gamma)(1/2-p)(1-(1\vee\gamma))})=o(1)$, for $1-(1\vee\gamma)\leqslant 0$, $\gamma\geqslant 0$, and $x/\varDelta_{v,\gamma}=o((\mathbf{D}T)^{2\nu(\gamma)(1/2-p)(\gamma-(1\vee\gamma))/(1+2\gamma)})=o(1)$, for $\gamma-(1\vee\gamma)\leqslant 0$, $\gamma\geqslant 0$, if $\mathbf{D}T\to\infty$. Thus $L_\gamma(x)\to 0$ for all $x\geqslant 0$ such that $x=o((\mathbf{D}T)^{(1/2-p)v(\gamma)})$, $0\leqslant p<1/2$, as $v\to 1$ and $\beta^2\to\infty$. □

Proof of Theorem 3. The proof of Theorem 3 follows from Lemma 2.4 in [6], where relations (2.12)–(2.14) hold with $H=2^{1+\gamma}$, $\bar{\varDelta}=\varDelta_v$. □

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REZIUMĖ

Atsitiktinio dėmenų skaičiaus sumoms didžiųjų nuokrypių diskontavimo versija

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Šiame darbe nagrinėjame sudėtinį atsitiktinį dydį $Z = \sum_{j=1}^{N} v^j X_j$, čia $0 < v < 1$, $Z = 0$, jei $N = 0$. Laikoma, kad nepriklausomi vienodai pasiskirstę atsitiktiniai dydžiai X_j, $j = 1, 2, \ldots$, turintys vidurkius $EX = \mu$ ir dispersijas $\mathbf{D}X = \sigma^2 > 0$, yra nepriklausomi nuo neneigiamas sveikas reikšmes įgyjančio atsitiktinio dydžio N. Pažymėtina, kad tokioje sumavimo schemoje mes turime nagrinėti du atvejus: $\mu \neq 0$ ir $\mu = 0$. Šis darbas yra skirtas sumos $\tilde{Z} = (Z - \mathbf{E}Z)(\mathbf{D}Z)^{-1/2}$ pasiskirstymo funkcijos aproksimacijos normaliąja pasiskirstymo funkcija viršutiniams įverčiams, didžiųjų nuokrypių teoremoms tiek Kramerio, tiek laipsninėse Liniko zonose ir tikimybės $\mathbf{P}(\tilde{Z} \geqslant x)$ eksponentinėms nelygybėms gauti.

Raktiniai žodžiai: kumuliantai, didžiųjų nuokrypių teoremos, diskontavimo ribinės teoremos, normalioji aproksimacija, atsitiktinio dėmenų skaičiaus sumos.