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# Joint universality of some zeta-functions. I

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**Abstract.** In the paper, the joint universality for the Riemann zeta-function and a collection of periodic Hurwitz zeta functions is discussed and basic results are given.

**Keywords:** joint universality, limit theorem, periodic Hurwitz zeta-function, Riemann zeta-function, space of analytic functions.

Let $\mathfrak{a} = \{a_m\colon m \in \mathbb{N}_0\}$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, be a periodic with minimal period $k \in \mathbb{N}$ sequence of complex numbers, and $\alpha$, $0 < \alpha \leqslant 1$, be a fixed number. The periodic Hurwitz zeta-function $\zeta(s, \alpha; \mathfrak{a})$, $s = \sigma + it$, is defined, for $\sigma > 1$, by

$$\zeta(s, \alpha; \mathfrak{a}) = \sum_{m=0}^{\infty} \frac{a_m}{(m+\alpha)^s},$$

and by analytic continuation elsewhere. If

$$a = \frac{1}{k} \sum_{l=0}^{k-1} a_l = 0,$$

then $\zeta(s, \alpha; \mathfrak{a})$ is an entire function. If $a \neq 0$, then the point $s = 1$ is a simple pole with residue $a$.

The universality of the function $\zeta(s, \alpha; \mathfrak{a})$ with transcendental parameter $\alpha$ has been obtained in [2]. Let $K$ be a compact subset of the strip $D = \{s \in \mathbb{C}\colon \frac{1}{2} < \sigma < 1\}$ with connected complement, and let $f(s)$ be a continuous function on $K$ which is analytic in interior of $K$. Then, for every $\varepsilon > 0$,

$$\liminf_{T \to \infty} \frac{1}{T} \mathrm{meas}\Big\{\tau \in [0, T]\colon \sup_{s \in K} \big|\zeta(s + i\tau, \alpha; \mathfrak{a}) - f(s)\big| < \varepsilon\Big\} > 0.$$

Here and in the sequel, $\mathrm{meas}\{A\}$ denotes the Lebesgue measure of a measurable set $A \subset \mathbb{R}$.

The joint universality of periodic Hurwitz zeta-functions was considered in a series of papers [5, 6, 7, 3, 9] and [8]. The most general result in this field is contained in [8]. For $j = 1, \dots, r$, let $\alpha_j$, $0 < \alpha_j \leqslant 1$, be fixed parameter, and $l_j \in \mathbb{N}$. Moreover, for $j = 1, \dots, r$ and $l = 1, \dots, l_j$, let $\mathfrak{a}_{jl} = \{a_{mjl}\colon m \in \mathbb{N}_0\}$ be a periodic with minimal period

$k_{jl} \in \mathbb{N}$ sequence of complex numbers, and $\zeta(s, \alpha_j; \mathfrak{a}_{jl})$ be corresponding periodic Hurwitz zeta-function. Define

$$L(\alpha_1, \dots, \alpha_r) = \{ \log(m + \alpha_j) \colon m \in \mathbb{N}_0,\ j = 1, \dots, r \}.$$

Moreover, let $k_j$ be the least common multiple of the periods $k_{j1}, k_{j2}, \dots, k_{jl_j}$, $j = 1, \dots, r$, and

$$B_j = \begin{pmatrix} a_{1j1} & a_{1j2} & \dots & a_{1jl_j} \\ a_{2j1} & a_{2j2} & \dots & a_{2jl_j} \\ \dots & \dots & \dots & \dots \\ a_{k_j j1} & a_{k_j j2} & \dots & a_{k_j jl_j} \end{pmatrix}, \quad j = 1, \dots, r.$$

**Theorem 1.** *(See [8].) Suppose that the set $L(\alpha_1, \dots, \alpha_r)$ is linearly independent over the field of rational numbers $\mathbb{Q}$ and that $\operatorname{rank}(B_j) = l_j$, $j = 1, \dots, r$. For every $j = 1, \dots, r$ and $l = 1, \dots, l_j$, let $K_{jl}$ be a compact subset of the strip $D = \{s \in \mathbb{C}\colon \frac{1}{2} < \sigma < 1\}$ with connected complement, and let $f_{jl}(s)$ be a continuous on $K_{jl}$ function which is analytic in interior of $K_{jl}$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{ \tau \in [0, T] : \sup_{1\leqslant j\leqslant r} \sup_{1\leqslant l\leqslant l_j} \sup_{s\in K_{jl}} \big| \zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s) \big| < \varepsilon \Big\} > 0.$$

The aim of this note is to give basics for the proof of the joint universality of the functions $\zeta(s)$ and $\zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl})$, $j = 1, \dots, r$, $l = 1, \dots, l_j$. Here, as usual, $\zeta(s)$ denotes the Riemann zeta-function, that is $\zeta(s) = \zeta(s, 1; \mathfrak{a}_1)$ with $\mathfrak{a}_1 = \{a_m = 1\colon m \in \mathbb{N}_0\}$.

**Theorem 2.** *Suppose that $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, and that other hypotheses of Theorem 1 hold. Moreover, let $K$ be a compact subset of the strip $D$ with connected complement, and let $f(s)$ be a continuous non-vanishing on $K$ function which is analytic in interior of $K$. Then, for every $\varepsilon > 0$,*

$$\liminf_{T\to\infty} \frac{1}{T} \operatorname{meas}\Big\{ \tau \in [0, T]\colon \sup_{s\in K} \big| \zeta(s + i\tau) - f(s) \big| < \varepsilon,$$
$$\sup_{1\leqslant j\leqslant r} \sup_{1\leqslant l\leqslant l_j} \sup_{s\in K_{jl}} \big| \zeta(s + i\tau, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s) \big| < \varepsilon \Big\} > 0.$$

The proof of Theorem 2 is based on a joint limit theorem in the space of analytic functions for the functions $\zeta(s)$ and $\zeta(s, \alpha_j; \mathfrak{a}_{jl})$, $j = 1, \dots, r$, $l = 1, \dots, l_j$.

Denote by $H(D)$ the space of analytic on $D$ functions equipped with the topology of uniform convergence on compacta, and let

$$H^\kappa(D) = \underbrace{H(D) \times \dots \times H(D)}_{\kappa},$$

where

$$\kappa = \sum_{j=1}^{r} l_j + 1.$$

Moreover, let $\gamma = \{s \in \mathbb{C}\colon |s| = 1\}$ be the unit circle on the complex plane. Define

$$\hat{\Omega} = \prod_p \gamma_p \quad \text{and} \quad \Omega = \prod_{m=0}^{\infty} \gamma_m,$$

where $\gamma_p = \gamma$ and $\gamma_m = \gamma$ for all primes $p$ and all $m \in \mathbb{N}_0$, respectively. Then, by the Tikhonov theorem, the tori $\hat{\Omega}$ and $\Omega$ are compact topological groups. Denote by $\mathcal{B}(S)$ the class of Borel sets of a space $S$. Then we obtain two probability spaces $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}), \hat{m}_H)$ and $(\Omega, \mathcal{B}(\Omega), m_H)$, where $\hat{m}_H$ and $m_H$ are probability measures on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$ and $(\Omega, \mathcal{B}(\Omega))$, respectively. Now let

$$\Omega^{r+1} = \hat{\Omega} \times \Omega_1 \times \cdots \times \Omega_r,$$

where $\Omega_j = \Omega$ for $j = 1, \ldots, r$. By the Tikhonov theorem again, $\Omega^{r+1}$ is a compact topological Abelian group, and this leads to the probability space $(\Omega^{r+1}, \mathcal{B}(\Omega^{r+1}), m_H^{r+1})$, where $m_H^{r+1}$ is the probability Haar measure on $(\Omega^{r+1}, \mathcal{B}(\Omega^{r+1}))$. Denote by $\hat{\omega}(p)$ the projection of $\hat{\omega} \in \hat{\Omega}$ to $\gamma_p$, and by $\omega(m)$ the projection of $\omega \in \Omega$ to $\gamma_m$. For brevity, let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_r)$, $\underline{\mathfrak{a}} = (\mathfrak{a}_{11}, \ldots, \mathfrak{a}_{1l_1}, \ldots, \mathfrak{a}_{r1}, \ldots, \mathfrak{a}_{rl_r})$, and let $\underline{\omega} = (\hat{\omega}, \omega_1, \ldots, \omega_r)$ be an element of $\Omega^{r+1}$. On the probability space $(\Omega^{r+1}, \mathcal{B}(\Omega^{r+1}), m_H^{r+1})$, define the $H^\kappa(D)$-valued random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$ by the formula

$$\begin{aligned}\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) = \big(&\zeta(s, \hat{\omega}), \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \ldots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}) \ldots, \\ &\zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \ldots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big),\end{aligned}$$

where

$$\zeta(s, \hat{\omega}) = \prod_p \left(1 - \frac{\hat{\omega}(p)}{p^s}\right)^{-1}$$

and

$$\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}\omega_j(m)}{(m + \alpha_j)^s}, \quad j = 1, \ldots, r,\ l = 1, \ldots, l_j.$$

Denote by $P_{\underline{\zeta}}$ the distribution of the random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$, and let $\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}}) = (\zeta(s), \zeta(s, \alpha_1; \mathfrak{a}_{11}), \ldots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \ldots, \zeta(s, \alpha_r; \mathfrak{a}_{r1}), \ldots, \zeta(s, \alpha_r; \mathfrak{a}_{rl_r}))$

**Theorem 3.** *Suppose that $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$. Then the probability measure*

$$P_T(A) \stackrel{\text{def}}{=} \frac{1}{T}\text{meas}\big\{\tau \in [0, T]\colon \underline{\zeta}(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}) \in A\big\}, \quad A \in \mathcal{B}\big(H^\kappa(D)\big),$$

*converges weakly to $P_{\underline{\zeta}}$ as $T \to \infty$.*

Taking into account a limited size of this note, we give only a sketch of the proof of Theorem 3. The proof of Theorem 2 as well as full proof of Theorem 3 will be given elsewhere.

Denote by $\mathcal{P}$ the set of all prime numbers.

1. Since the numbers $\alpha_1, \ldots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, we have that the set

$$L \stackrel{\text{def}}{=} \big\{(\log p\colon p \in \mathcal{P}), \big(\log(m + \alpha_j)\colon m \in \mathbb{N}_0,\ j = 1, \ldots, r\big)\big\}$$

is linearly independent over $\mathbb{Q}$. Consider the probability measure

$$Q_T(A) = \frac{1}{T}\text{meas}\big\{\tau \in [0,T]\colon \big((p^{-i\tau}\colon p \in \mathcal{P}), \big((m+\alpha_1)^{-i\tau}\colon m \in \mathbb{N}_0\big), \dots, \\ \big((m+\alpha_r)^{-i\tau}\colon m \in \mathbb{N}_0\big)\big) \in A\big\}, \quad A \in \mathcal{B}\big(\Omega^{r+1}\big).$$

Then, using the above remark on the set $L$ and applying the Fourier transform method, we find that the measure $Q_T$ converges weakly to the Haar measure $m_H^{r+1}$ as $T \to \infty$.

2. Let $\sigma_1 > \frac{1}{2}$ be a fixed number, and

$$v_n(m) = \exp\bigg\{-\bigg(\frac{m}{n}\bigg)^{\sigma_1}\bigg\}, \quad m, n \in \mathbb{N},$$
$$v_n(m,\alpha_j) = \exp\bigg\{-\bigg(\frac{m+\alpha_j}{n+\alpha_j}\bigg)^{\sigma_1}\bigg\}, \quad m, n \in \mathbb{N}_0,\ j = 1,\dots,r.$$

Then by a standard way can be proved that the series

$$\zeta_n(s) = \sum_{m=1}^{\infty} \frac{v_n(m)}{m^s}, \qquad \zeta_n(s,\alpha_j;\mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl} v_n(m,\alpha_j)}{(m+\alpha_j)^s}, \quad j = 1,\dots,r,$$

are absolutely convergent for $\sigma > \frac{1}{2}$. For $m \in \mathbb{N}$, define

$$\hat{\omega}(m) = \prod_{p^l \| m} \hat{\omega}^l(p),$$

where $p^l \| m$ means that $p^l \mid m$ but $p^{l+1} \nmid m$, and let

$$\zeta_n(s,\hat{\omega}) = \sum_{m=1}^{\infty} \frac{v_n(m)\hat{\omega}(m)}{m^s},$$
$$\zeta_n(s,\alpha_j,\omega_j;\mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}\omega_j(m) v_n(m,\alpha_j)}{(m+\alpha_j)^s}, \quad j = 1,\dots,r.$$

The latter series, clearly, also are absolutely convergent for $\sigma > \frac{1}{2}$. Let, for brevity,

$$\underline{\zeta}_n(s,\underline{\alpha};\underline{\mathfrak{a}}) = \big(\zeta_n(s), \zeta_n(s,\alpha_1;\mathfrak{a}_{11}), \dots, \zeta_n(s,\alpha_1;\mathfrak{a}_{1l_1}), \dots, \\ \zeta_n(s,\alpha_r;\mathfrak{a}_{r1}), \dots, \zeta_n(s,\alpha_r;\mathfrak{a}_{rl_r})\big),$$

and

$$\underline{\zeta}_n(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}}) = \big(\zeta_n(s,\hat{\omega}), \zeta_n(s,\alpha_1,\omega_1;\mathfrak{a}_{11}), \dots, \zeta_n(s,\alpha_1,\omega_1;\mathfrak{a}_{1l_1}), \dots, \\ \zeta_n(s,\alpha_r,\omega_r;\mathfrak{a}_{r1}), \dots, \zeta_n(s,\alpha_r,\omega_r;\mathfrak{a}_{rl_r})\big).$$

Then the next step of the proof of Theorem 3 consists of the proof that the probability measures

$$\frac{1}{T}\text{meas}\big\{\tau \in [0,T]\colon \underline{\zeta}_n(s+i\tau,\underline{\alpha};\underline{\mathfrak{a}}) \in A\big\}, \quad A \in \mathcal{B}\big(H^\kappa(D)\big),$$

and

$$\frac{1}{T}\text{meas}\big\{\tau \in [0,T]\colon \underline{\zeta}_n(s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}}) \in A\big\}, \quad A \in \mathcal{B}\big(H^\kappa(D)\big),$$

both converge weakly to the same probability measure $P$ on $(H^\kappa(D),\mathcal{B}(H^\kappa(D)))$ as $T \to \infty$. For this, the weak convergence of measure $Q_T$ to $m_H^{r+1}$ as well as the invariance of $m_H^{r+1}$ and properties of weak convergence of probability measures are applied.

3. Now we approximate in the mean $\zeta(s,\underline{\alpha};\underline{\mathfrak{a}})$ by $\zeta_n(s,\underline{\alpha};\underline{\mathfrak{a}})$ and $\zeta(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})$ by $\zeta_n(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})$.

Let $\{K_m\colon m \in \mathbb{N}\}$ be a sequence of compact subsets of the strip $D$ such that

$$\bigcup_{m=1}^{\infty} K_m = D,$$

$K_m \subset K_{m+1}$ for all $m \in \mathbb{N}$, and, for every compact $K \subset D$, there exists $m$ such that $K \subset K_m$. For $f, g \in H(D)$, let

$$\rho(f,g) = \sum_{m=1}^{\infty} 2^{-m} \frac{\sup_{s\in K_m} |f(s)-g(s)|}{1+\sup_{s\in K_m} |f(s)-g(s)|}.$$

Then $\rho$ is a metric on $H(D)$ which induces its topology of uniform convergence on compacta. Now if $\underline{f} = (f_0, f_{11}, \dots, f_{1l_1}, \dots, f_{r1}, \dots, f_{rl_r})$, $\underline{g} = (g_0, g_{11}, \dots, g_{1l_1}, \dots, g_{r1}, \dots, g_{rl_r}) \in H^\kappa(D)$, and

$$\rho_\kappa(\underline{f},\underline{g}) = \max\Big(\max_{1\leqslant j\leqslant r}\max_{1\leqslant l\leqslant l_j} \rho(f_{jl},g_{jl}), \rho(f_0,g_0)\Big),$$

then $\rho_\kappa$ is a metric on $H^\kappa(D)$ with induces its topology of uniform convergence on compacta.

In this step, we prove the following equalities:

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho_\kappa\big(\underline{\zeta}(s+i\tau,\underline{\alpha};\underline{\mathfrak{a}}), \underline{\zeta}_n(s+i\tau,\underline{\alpha};\underline{\mathfrak{a}})\big)\, d\tau = 0, \tag{1}$$

and if $\alpha_1,\dots,\alpha_r$ are algebraically independent over $\mathbb{Q}$, then, for almost all $\underline{\omega} \in \Omega^{r+1}$,

$$\lim_{n\to\infty}\limsup_{T\to\infty}\frac{1}{T}\int_0^T \rho_\kappa\big(\underline{\zeta}(s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}}), \underline{\zeta}_n(s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})\big)\, d\tau = 0. \tag{2}$$

The proof of the above equalities easily follows from their one-dimensional versions in [2] and [4].

4. Additionally to $P_T$, define one more probability measure

$$\hat{P}_T(A) = \frac{1}{T}\text{meas}\{\tau \in [0,T]\colon \underline{\zeta}(s+i\tau,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}}) \in A\}, \quad A \in \mathcal{B}\big(H^\kappa(D)\big).$$

Using the limit theorems stated in Step 2, the approximation in the mean (1) and (2) as well as Theorem 4.2 from [1], we prove that both the measures $P_T$ and $\hat{P}_T$ converge weakly to the same probability measure $P$ on $(H^\kappa(D),\mathcal{B}(H^\kappa(D)))$ as $T \to \infty$.

5. It remains to show that $P = P_{\underline{\zeta}}$. For this, the ergodicity of the group of transformations $\{\Phi_\tau\colon \tau \in \mathbb{R}\}$ on $\Omega^{r+1}$ defined by $\Phi_\tau(\underline{\omega}) = a_\tau\underline{\omega}$, $\underline{\omega} \in \Omega^{r+1}$, where

$$a_\tau = \big\{\big(p^{-i\tau}\colon p \in \mathcal{P}\big), \big((m+\alpha_1)^{-i\tau}\colon m \in \mathbb{N}_0\big), \dots, \big((m+\alpha_r)^{-i\tau}\colon m \in \mathbb{N}_0\big)\big\}, \quad \tau \in \mathbb{R},$$

as well as the classical Birkhoff–Khinchine theorem is applied.

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REZIUMĖ

**Keleto dzeta funkcijų jungtinis universalumas. I**

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Straipsnyje nagrinėjamas Rymano dzeta funkcijos ir periodinių Hurvico dzeta funkcijų rinkinio jungtinis universalumas.

*Raktiniai žodžiai*: analizinių funkcijų erdvė, jungtinis universalumas, periodinė Hurvico dzeta-funkcija, ribinė teorema, Rymano dzeta funkcija.