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Applying the IR statistic to estimate the Hurst index of the fractional geometric Brownian motion

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Abstract. In 2010 J.M. Bardet and D. Surgailis [1] have introduced the increment ratio (IR) statistic which measures the roughness of random paths. It was shown that this statistic was applicable in the cases of diffusion processes driven by the standard Brownian motion, certain Gaussian processes and the Lévy process. This paper shows that the IR statistic can be applied to estimate the Hurst index H of the fractional geometric Brownian motion.

Keywords: increment ratio statistic, fractional geometric Brownian motion, fractional Black–Scholes model, Hurst index estimation.

1 Introduction

The IR statistic is defined as

$$R^{p,n}(f)=\frac{1}{n-p}\sum_{k=0}^{n-p-1}\frac{|\Delta_k^{p,n}f+\Delta_{k+1}^{p,n}f|}{|\Delta_k^{p,n}f|+|\Delta_{k+1}^{p,n}f|}$$

where $\Delta_k^{p,n}f$ denotes the p-order increment of a real-valued function f at t_k^n, $p=1,2,\dots$, $k=0,1,\dots,n-p$, that is,

$$\Delta_k^{1,n}f=f\bigl(t_{k+1}^n\bigr)-f\bigl(t_k^n\bigr),\qquad \Delta_k^{p,n}f=\Delta_k^{1,n}\Delta_k^{p-1,n}f.$$

J.M. Bardet and D. Surgailis showed that if X is a fractional Brownian motion (B^H) with parameter $H\in(0,1)$ then

$$R^{p,n}(f)\xrightarrow{\text{a.s.}}\Lambda_p(H)\quad\text{as } n\to\infty,\ p=1,2 \tag{1}$$

where

$$\Lambda_p(H)=\mathbf{E}\frac{|\Delta_0^pB^H+\Delta_1^pB^H|}{|\Delta_0^pB^H|+|\Delta_1^pB^H|}.$$

The $R^{2,n}(f)$ statistic is suited better for practical purposes than $R^{1,n}(f)$ since the error arising from approximating $\Lambda_2(H)$ with a line is considerably smaller than that of $\Lambda_1(H)$. Fig. 1 presents the graph of $\Lambda_2(H)$ as well as the graph of $R^{2,100}(B^H)$ averaged over 50 sample paths.

In the recent years it has been proposed by several authors to replace the classic Black–Scholes model, based on the standard geometric Brownian motion, with its fractional counterpart. This would enable the model to handle the possible existence of long-range dependance in the observed data. In this paper it is shown that the convergence (1) holds for $H \in (1/2; 7/8)$ when $p = 1$ and $H \in (1/2; 1)$ when $p = 2$ if X is the solution of the fractional Black–Scholes equation.

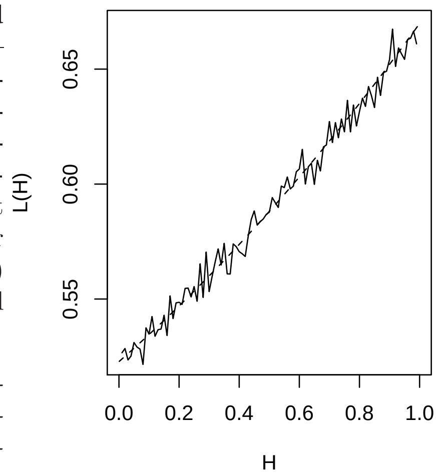

Fig. 1: Graphs of $\Lambda_2(H)$ and $R^{2,100}(B^H)$.

Theorem 1. *Let $B^H = \{B_t^H,\ t \in [0,1]\}$ denote the fractional Brownian motion with parameter H and let X be the solution of the fractional Black–Scholes equation*

$$dX_t = \mu X_t\,dt + \sigma X_t\,dB_t^H, \quad \mu, \sigma, X_0 \in \mathbb{R} \quad (2)$$

observed at times $t_k^n = \frac{k}{2^n}$, $k = 0, 1, \dots, 2^n$. Then

$$R^{p,n}(X) \xrightarrow{\text{a.s.}} \Lambda_p(H) \quad \text{as } n \to \infty,\ p = 1, 2$$

for $H \in (1/2; 7/8)$ when $p = 1$ and $H \in (1/2; 1)$ when $p = 2$.

The proof of this theorem is based on the following lemma which is a generalization of the corresponding lemma in the paper of J.M. Bardet and D. Surgailis.

Lemma 1. *Let $\psi(x_1, x_2) = \frac{|x_1+x_2|}{|x_1|+|x_2|}$, $x_1, x_2 \in \mathbb{R}$, and let (Z_1, Z_2) be a Gaussian vector with zero mean and dispersion $\mathbf{E}Z_i^2 = 1$, $i = 1, 2$. Then for any r.v. ξ_i, $i = 1, 2$,*

$$\mathbf{E}\big|\psi(Z_1 + \xi_1, Z_2 + \xi_2) - \psi(Z_1, Z_2)\big| \leqslant 16 \max_{i=1,2} \sqrt[3]{\mathbf{E}\xi_i^2}, \quad k \geqslant 1.$$

2 Proofs

Proof of Lemma 1. Let $\delta^2 = \max_{i=1,2} \mathbf{E}\xi_i^2$. Denote $U := \psi(Z_1 + \xi_1, Z_2 + \xi_2) - \psi(Z_1, Z_2) = U_\delta + U_\delta^c$,

$$U_\delta := U\mathbf{1}_{A_\delta} = \big(\psi(Z_1 + \xi_1, Z_2 + \xi_2) - \psi(Z_1, Z_2)\big)\mathbf{1}_{A_\delta},$$
$$U_\delta^c := U\mathbf{1}_{A_\delta^c} = \big(\psi(Z_1 + \xi_1, Z_2 + \xi_2) - \psi(Z_1, Z_2)\big)\mathbf{1}_{A_\delta^c},$$

where $\mathbf{1}_{A_\delta}$ is the indicator of the event

$$A_\delta := \big\{|Z_1| > \delta^{2/3},\ |Z_2| > \delta^{2/3},\ |\xi_1| < \delta^{2/3}/2,\ |\xi_2| < \delta^{2/3}/2\big\}$$

and $\mathbf{1}_{A_\delta^c} = 1 - \mathbf{1}_{A_\delta}$ is the indicator of the complementary event A_δ^c. Clearly,

$$\begin{aligned}\mathbf{E}|U_\delta^c| \leqslant{}& 2\big[\mathbf{P}\big(|Z_1| < \delta^{2/3}\big) + \mathbf{P}\big(|Z_2| < \delta^{2/3}\big) + \mathbf{P}\big(|\xi_1| \geqslant \delta^{2/3}/2\big) + \mathbf{P}\big(|\xi_2| \geqslant \delta^{2/3}/2\big)\big] \\ \leqslant{}& \frac{8}{\sqrt{2\pi}}\,\delta^{2/3} + 8 \max_{i=1,2} \frac{\mathbf{E}|\xi_i|^2}{\delta^{4/3}} \leqslant 12\delta^{2/3}.\end{aligned}$$

It remains to estimate $\mathbf{E}|U_\delta|$. By the mean value theorem,

$$U_\delta = \left(\xi_1 \frac{\partial\psi}{\partial x_1}(Z_1+\theta\xi_1, Z_2+\theta\xi_2) + \xi_2 \frac{\partial\psi}{\partial x_2}(Z_1+\theta\xi_1, Z_2+\theta\xi_2)\right)\mathbf{1}_{A_\delta}, \quad 0<\theta(\omega)<1,$$

where

$$\left|\frac{\partial\psi}{\partial x_i}(x_1,x_2)\right| = \frac{|(|x_1|+|x_2|)\operatorname{sgn}(x_1+x_2) - |x_1+x_2|\operatorname{sgn}(x_i)|}{(|x_1|+|x_2|)^2} \leqslant \frac{2}{|x_1|+|x_2|}.$$

Thus

$$\begin{aligned}\left|\frac{\partial\psi}{\partial x_i}(Z_1+\theta\xi_1, Z_2+\theta\xi_2)\right|\mathbf{1}_{A_\delta} &\leqslant \frac{2}{|Z_1+\theta\xi_1|+|Z_2+\theta\xi_2|}\mathbf{1}_{A_\delta}\\ &\leqslant \frac{2}{|Z_1+\theta\xi_1|+|Z_2+\theta\xi_2|}\mathbf{1}_{B_\delta} \leqslant \frac{2}{\delta^{2/3}}\mathbf{1}_{B_\delta},\end{aligned}$$

where

$$B_\delta = \left\{|Z_1+\theta\xi_1| > \delta^{2/3}/2,\ |Z_1+\theta\xi_1| > \delta^{2/3}/2,\ |\xi_1| \leqslant \delta^{2/3}/2,\ |\xi_2| \leqslant \delta^{2/3}/2\right\}.$$

Therefore

$$\begin{aligned}\mathbf{E}|U_\delta| \leqslant{}& \mathbf{E}^{1/2}\xi_1^2 \cdot \mathbf{E}^{1/2}\left[\left|\frac{\partial\psi}{\partial x_1}(Z_1+\theta\xi_1, Z_2+\theta\xi_2)\right|^2 \mathbf{1}_{A_\delta}\right]\\ &+ \mathbf{E}^{1/2}\xi_2^2 \cdot \mathbf{E}^{1/2}\left[\left|\frac{\partial\psi}{\partial x_2}(Z_1+\theta\xi_1, Z_2+\theta\xi_2)\right|^2 \mathbf{1}_{A_\delta}\right] \leqslant 4\delta^{2/3}\end{aligned}$$

and

$$\mathbf{E}|U| \leqslant 16\delta^{2/3}.$$

Proof of Theorem 1. Let Δt^n denote the mesh of the subdivision, that is, $\Delta t^n := 2^{-n}$. The solution of the Black–Scholes equation (2) is $X_t = c\exp(\mu t + \sigma B_t^H)$. Therefore $R^{1,n}(X)$ can be rewritten as

$$R^{1,n}(X) = \frac{1}{2^n-1}\sum_{k=0}^{2^n-2} \frac{|\exp\{\mu\Delta t^n + \sigma\Delta B_{k+1}^{H,n}\} - \exp\{-\mu\Delta t^n - \sigma\Delta B_k^{H,n}\}|}{|1-\exp\{-\mu\Delta t^n - \sigma\Delta B_k^{H,n}\}| + |\exp\{\mu\Delta t^n + \sigma\Delta B_{k+1}^{H,n}\} - 1|}.$$

For briefness, let the index n be omitted. Then the Taylor expansion yields

$$\begin{gathered}\exp\left\{\mu\Delta t + \sigma\Delta B_{k+1}^H\right\} - 1 = \sigma\Delta B_{k+1}^H + \left(\mu\Delta t + R\left(\mu\Delta t + \sigma\Delta B_{k+1}^H\right)\right),\\ 1-\exp\left\{-\mu\Delta t - \sigma\Delta B_k^H\right\} = \sigma\Delta B_k^H + \left(\mu\Delta t - R\left(-\mu\Delta t - \sigma\Delta B_k^H\right)\right),\end{gathered}$$

$$\begin{aligned}&\exp\left\{\mu\Delta t + \sigma\Delta B_{k+1}^H\right\} - \exp\left\{-\mu\Delta t - \sigma\Delta B_k^H\right\}\\ &\quad= \sigma\left(\Delta B_{k+1}^H + \Delta B_k^H\right) + \left(2\mu\Delta t + R\left(\mu\Delta t + \sigma\Delta B_{k+1}^H\right) - R\left(-\mu\Delta t - \sigma\Delta B_k^H\right)\right),\end{aligned}$$

where

$$R(x) = \frac{x^2}{2}e^{\theta x}, \quad 0<\theta<1.$$

Let

$$Z_1(k) = \sigma \Delta B_k^H, \qquad \xi_1(k) = \mu\Delta t - R\big(-\mu\Delta t - \sigma\Delta B_k^H\big),$$
$$Z_2(k) = \sigma \Delta B_{k+1}^H, \qquad \xi_2(k) = \mu\Delta t + R\big(\mu\Delta t + \sigma\Delta B_{k+1}^H\big).$$

Obviously

$$R^{1,n}(X) = \frac{1}{2^n-1}\sum_{k=0}^{2^n-2}\frac{|Z_1(k)+\xi_1(k)+Z_2(k)+\xi_2(k)|}{|Z_1(k)+\xi_1(k)|+|Z_2(k)+\xi_2(k)|}.$$

Define $\psi_k := \mu\Delta t + \sigma\Delta B_k^H$. Then

$$\begin{aligned}\mathbf{E}(\xi_1(k))^2 &\leqslant 2(\mu\Delta t)^2 + \mathbf{E}\big(\psi_k^4 e^{-2\theta\psi_k}\big) \leqslant 2(\mu\Delta t)^2 + \sqrt{105(\mathbf{E}\psi_k^2)^4\cdot \mathbf{E}e^{-4\theta\psi_k}}\\ &\leqslant 4\sqrt{105}\big[(\mu\Delta t)^4 + \sigma^4(\Delta t)^{4H}\big]\sqrt{\mathbf{E}e^{-4\theta\psi_k}}\end{aligned}$$

since ψ_k are Gaussian and therefore $\mathbf{E}\psi_k^8 = 105(\mathbf{E}\psi_k^2)^4$. Further,

$$\mathbf{E}e^{-4\theta\psi_k} \leqslant \mathbf{E}e^{4|\psi_k|} \leqslant e^{4|\mu|\Delta t}\mathbf{E}e^{4|\sigma\Delta B_k^H|} \leqslant 2e^{4|\mu|+8\sigma^2},$$

which yields that

$$\begin{aligned}\mathbf{E}(\xi_1(k))^2 &\leqslant 4\sqrt{210}\big[(\mu\Delta t)^4 + \sigma^4(\Delta t)^{4H}\big]e^{2|\mu|+4\sigma^2}\\ &= 4\sqrt{210}(\Delta t)^2\big[\mu^4(\Delta t)^2 + \sigma^4(\Delta t)^{2(2H-1)}\big]e^{2|\mu|+4\sigma^2}.\end{aligned}$$

Since $2H-1>0$, we get that $\mathbf{E}(\xi_1(k))^2 = \mathcal{O}(\Delta t)^2$. Similarly, $\mathbf{E}(\xi_2(k))^2 = \mathcal{O}(\Delta t)^2$ and according to Lemma 1

$$\begin{aligned}&\mathbf{E}\bigg|\frac{|Z_1(k)+\xi_1(k)+Z_2(k)+\xi_2(k)|}{|Z_1(k)+\xi_1(k)|+|Z_2(k)+\xi_2(k)|} - \frac{|Z_1(k)+Z_2(k)|}{|Z_1(k)|+|Z_2(k)|}\bigg| =\\ &= \mathbf{E}\bigg|\frac{|\Delta X_k+\Delta X_{k+1}|}{|\Delta X_k|+|\Delta X_{k+1}|} - \frac{|\Delta B_k^H+\Delta B_{k+1}^H|}{|\Delta B_k^H|+|\Delta B_{k+1}^H|}\bigg| = \mathcal{O}(2^{-n})^{2/3}.\end{aligned}$$

Then

$$\begin{aligned}\mathbf{E}\big|R^{1,n}(X) - R^{1,n}(B^H)\big| &\leqslant \frac{1}{n-1}\sum_{k=0}^{n-2}\mathbf{E}\bigg|\frac{|\Delta X_k+\Delta X_{k+1}|}{|\Delta X_k|+|\Delta X_{k+1}|} - \frac{|\Delta B_k^H+\Delta B_{k+1}^H|}{|B_k^H|+|\Delta B_{k+1}^H|}\bigg|\\ &= \mathcal{O}(2^{-n})^{2/3}\end{aligned}$$

and, consequently, $R^{1,n}(X) \xrightarrow{\text{P}} R^{1,n}(B^H)$, $n\to\infty$. Let $\zeta_n := R^{1,n}(X) - R^{1,n}(B^H)$. Then the Chebyshev's inequality yields

$$\mathbf{P}\big(|\zeta_n| > 2^{-n/3}\big) \leqslant 2^{n/3}\mathbf{E}|\zeta_n| \leqslant 2^{-n/3}$$

and

$$\sum_{n=1}^{\infty}\mathbf{P}\big(|\zeta_n| > 2^{-n/3}\big) \leqslant \sum_{n=1}^{\infty}2^{-n/3} < \infty.$$

According to the Borel–Cantelli lemma,

$$\mathbf{P}\Big(\limsup_{n\to\infty}\big\{|\zeta_n| > 2^{-n/3}\big\}\Big) = 0$$

which implies that $R^{1,n}(X) \overset{\text{a.s.}}{\to} R^{1,n}(B^H)$, $n \to \infty$.

The convergence $R^{p,n}(B^H) \overset{\text{a.s.}}{\to} \Lambda_1(H)$, $n \to \infty$ is established in [1] and holds for $H \in (0;7/8)$ when $p = 1$ and $H \in (0;1)$ when $p = 2$. Clearly, provided $R^{1,n}(X) \overset{\text{a.s.}}{\to} R^{1,n}(B^H)$ and $R^{p,n}(B^H) \overset{\text{a.s.}}{\to} \Lambda_1(H)$, $n \to \infty$ it follows that $R^{1,n}(X) \overset{\text{a.s.}}{\to} \Lambda_1(H)$, $n \to \infty$ which completes the proof for the case $p = 1$. The proof for $p = 2$ follows analogously.

Fig. 2 presents the graph of $\Lambda_2(H)$ together with the graph of $R^{2,100}(X)$ averaged over 50 sample paths, X being the solution of the Black–Scholes equation. Table 1 shows the comparison of mean squared errors of $R^{2,n}(X) - \Lambda_2(H)$ and $R^{2,n}(B^H) - \Lambda_2(H)$ for the sample path lengths $n = 100, 200, 500$ and the numbers of sample paths $Nsp = 20$, 50, 100. The constants of the Black–Scholes process were chosen as $X_0 = 1$, $\mu = -0.3$, $\sigma = 0.5$. All computations were performed using the R software environment [2].

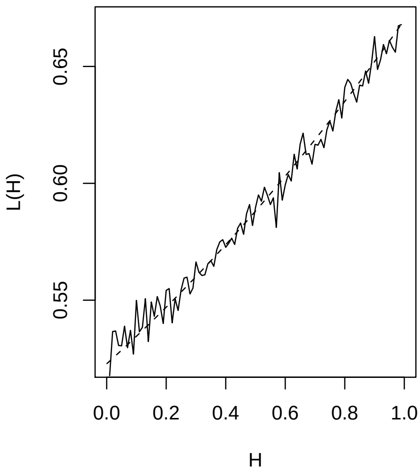

Fig. 2: Graphs of $\Lambda_2(H)$ and $R^{2,100}(X)$.

Table 1: MSE $\cdot 10^2$.

Nsp	n	100	500	1000
10	B^H	1.0997	0.6383	0.3592
	X	1.1000	0.5891	0.3916
50	B^H	0.5043	0.2077	0.1537
	X	0.5817	0.2300	0.1598
100	B^H	0.3421	0.1781	0.1246
	X	0.3398	0.1682	0.1257

References

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REZIUMĖ

IR statistikos taikymas trupmeninio geometrinio Brauno judesio Hursto indekso vertinimui

D. Melichov

2010 m. J.M. Bardet ir D. Surgailis [1] įvedė pokyčių santykio (IR) statistiką, kuri matuoja atsitiktinio proceso trajektorijų šiurkštumą. Jie parodė, kad ši statistika gali būti taikoma difuziniams procesams, valdomiems standartinio Brauno judesio, tam tikriems Gauso procesams ir Lévy procesams. Šiame straipsnyje įrodoma, kad IR statistika gali būti taikoma trupmeninio geometrinio Brauno judesio Hursto indekso H vertinimui.

Raktiniai žodžiai: pokyčių santykio statistika, trupmeninis Brauno judesys, trupmeninis Black–Scholes modelis, Hursto indekso vertinimas.