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On hamiltonicity of uniform random intersection graphs

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Abstract. We give a sufficient condition for the hamiltonicity of the uniform random intersection graph $G_{n,m,d}$. It is a graph on n vertices, where each vertex is assigned d keys drawn independently at random from a given set of m keys, and where any two vertices establish an edge whenever they share at least one common key. We show that with probability tending to 1 the graph $G_{n,m,d}$ has a Hamilton cycle provided that $n = 2^{-1}m(\ln m + \ln\ln m + \omega(m))$ with some $\omega(m) \to +\infty$ as $m \to \infty$.

Keywords: random graph, intersection graph, Hamilton cycle, clustering.

1 Introduction and results

A random intersection graph (RIG) on a vertex set $V = \{v_1, \dots, v_n\}$ is defined by the collection of random subsets $S_1, \dots, S_n$ of an auxiliary set W. Any two vertices v_i, v_j are connected by an edge in RIG whenever the sets $S(v_i)$ and $S(v_j)$ intersect. Elements of W are called attributes (or keys). In particular, one can interpret $S_i = S(v_i)$ as the attribute set characterizing the vertex v_i. The random sets are usually assumed to be independent, but not necessarily identically distributed.

The random intersection graph, where each vertex collects its attribute set by inserting attributes independently at random with a given probability was introduced by Karoński, Scheinerman and Singer-Cohen [13] and Singer-Cohen [17]. A more general model was considered by Godehard and Jaworski [11], see also Shang [16]. Random intersection graphs have received considerable attention in recent literature ([1, 2, 4, 5, 3, 6, 7, 8, 12, 18], etc.)

In the present note we consider a particular class of RIG, where the sets $S_1, \dots, S_n$ are of the same size, say d. In particular, every S_i is uniformly distributed in the class of all subsets of W of size d. Given m (the size of the auxiliary set W), n (the number of vertices), and d such a graph is denoted $G_{n,m,d}$ and called the uniform RIG. Interest to uniform RIG was motivated by the paper of Eshenauer and Gligor [9] who suggested a random key predistribution scheme, based on uniform RIG, that ensures the security of links in a wireless sensor network. Afterwards, the connectivity and component evolution of uniform RIG was shown in [2] and [5] (see also references therein). Recent papers [14] and [15] address the hamiltonicity and independence number.

We study the hamiltonicity of $G_{n,m,d}$ in the case where d is arbitrary, but remains fixed as $n, m \to \infty$. Observe that such a random graph has statistically dependent edges. Indeed, for any three vertices, say x, y, z we have

$$\liminf_{m\to\infty} \mathbf{P}(x \sim y,\, x \sim z,\, v \sim z \mid x \sim y,\, x \sim z) \geqslant d^{-1} > 0, \tag{1}$$

while the edge probability $p'_d = p'_d(n) = \mathbf{P}(x \sim y) = d^2m^{-1} + O(d^4m^{-2})$ tends to 0 as $m \to \infty$. Here $x \sim y$ denotes the adjacency relation. In what follows we say that an event holds with a high probability (whp for short) if its probability tends to 1.

Lemma 1 below gives a sufficient condition for the hamiltonicity of $G_{n,m,d}$ in terms of n and m (n should be sufficiently larger than m). A necessary condition is formulated in Lemma 2. These conditions refer to the same order

$$n = \Theta(m \ln m), \tag{2}$$

but the constants do not match. In particular, we show that for $n > (0.5 + \varepsilon)m \ln m$ the graph $G_{n,m,d}$ is Hamiltonian whp, while for $n < (d^{-2} - \varepsilon)m \ln m$ the graph $G_{n,m,d}$ has no Hamilton cycle whp as $m \to \infty$. Here $\varepsilon > 0$ is arbitrarily small.

Lemma 1. *Let $d \geqslant 2$ be an integer. Assume that for some $\omega(m) \to +\infty$ we have as $m \to \infty$*

$$n = 2^{-1}m\bigl(\ln m + \ln\ln m + \omega(m)\bigr). \tag{3}$$

Then whp $G_{n,m,d}$ is Hamiltonian as $n \to \infty$.

Lemma 2. *Let $d \geqslant 2$ be an integer. Assume that for some $\varepsilon \in (0, d^{-2})$ we have as $m \to \infty$*

$$n \leqslant (d^{-2} - \varepsilon)m \ln m. \tag{4}$$

Then the following statements hold true:

(i) *whp $G_{n,m,d}$ has a vertex of degree at most* 1,

(ii) *whp $G_{n,m,d}$ has no Hamilton cycle.*

Let us compare the Hamiltonicity threshold of the classical Erdős-Rényi graph $G(n,p)$, where edges are inserted *independently* with probability p, with our results for $G_{n,m,d}$. Recall that $G(n,p)$ has a Hamilton cycle with probability tending to 1 whenever

$$p = p(n) = n^{-1}\bigl(\ln(n) + \ln\ln(n) + \omega(n)\bigr) \tag{5}$$

for some $\omega(n) \to +\infty$, as $n \to \infty$, see, e.g. [6]. From Lemmas 1 and 2 we conclude that $G_{n,m,d}$ is Hamiltonian whp for $p'_2 = (2^{-1}d^2 + \varepsilon)n^{-1}\ln n$, and $G_{n,m,d}$ has no Hamilton cycle whp for $p'_2 = (1 - \varepsilon)n^{-1}\ln n$. Here $\varepsilon > 0$ is arbitrarily small.

Our sufficient condition (3) provides an improvement upon the corresponding condition shown in Theorem 2 of [14], $n \geqslant (1+\varepsilon)\binom{m}{d}\ln\bigl(\binom{m}{d}\bigr)$.

2 Proofs

Here we prove Lemmas 1 and 2. An auxiliary result used in the proof of Lemma 1 is stated separately in Lemma 3 at the end of the section.

Proof of Lemma 1. Given S_i, let $s_i \subset S_i$ be a random subset of size 2. Let G^* be a random multi-graph on the vertex set $W = \{w_1, \dots, w_m\}$ with the edge set

$\{s_1, \ldots, s_n\}$. Let n' be the number of distinct edges s_i and let $G' \subset G^*$ be the subgraph containing all but distinct edges. Observe that given n', the graph G' has the same distribution as the uniform Erdős-Rényi graph on m vertices having n' random edges (where the set of edges is uniformly distributed in the class of all subsets of size n' of the set of all possible pairs of vertices). It is known (see, e.g., [6]) that such an Erdős-Rényi graph contains a Hamilton cycle whp provided that

$$n' \geqslant 2^{-1} m\bigl(\ln m + \ln\ln m + \omega(m)\bigr) \tag{6}$$

for some $\omega(m) \to \infty$ as $m \to \infty$. It follows from (3), by Lemma 3, that (6) holds with a high probability. Therefore, whp the graph G' is Hamiltonian.

Now the lemma follows from the simple observation that the hamiltonicity of G' implies the hamiltonicity of $G = G_{n,m,d}$. Indeed, let $s_{i_1}, \ldots, s_{i_m}$ be (the edges of) a Hamilton cycle of G'. The corresponding vertices $v_{i_1}, \ldots, v_{i_m}$ build a cycle, say C, in G. Let V' denote the set of vertices outside C. Split $V' = V_1 \cup V_2 \cup \cdots \cup V_m$ into non-interecting classes of vertices such that, for every j, all vertices of V_j share the attribute $s_{i_j} \cap s_{i_{j+1}}$ (we define $s_{i_{m+1}} := s_{i_1}$). In particular vertices from V_j belong to a clique of G attached to the cycle C and containing vertices v_{i_j} and $v_{i_{j+1}}$. For all vertices of G are covered by the cycle C and several cliques attached to its edges we can extend C to a Hamilton cycle. $\square$

Proof of Lemma 2. Note that (i) implies (ii). We need to prove (i). In the proof we apply the following inequalities from [4]. Let S_1 and S_2 be two independent random sets uniformly distributed in the class of subsets of $\{1, \ldots, m\}$ of sizes x and y respectively. Then as $m \to \infty$

$$\mathbf{P}\bigl(|S_1 \cap S_2| \geqslant 1\bigr) = xym^{-1} + O\bigl(x^2y^2m^{-2}\bigr). \tag{7}$$

Given $v \in V$, let $\mathbb{I}_v$ denote the indicator of the event $\{d(v) \leqslant 1\}$. Hence, $\mathbb{I}_v = 1$ whenever $d(v) \leqslant 1$. Here $d(v)$ denotes the degree of v (the number of neighbours of v in $G_{n,m,d}$). Let $X = \sum_{v \in V} \mathbb{I}_v$ count vertices of degree at most 1. Note that (i) is equivalent to the limit $\mathbf{P}(X > 0) \to 1$ as $n \to \infty$. In the proof of this limit we show that

$$\lim_n \mathbf{E}X = +\infty \quad \text{and} \quad \sqrt{\mathbf{Var}X} = o(\mathbf{E}X) \tag{8}$$

and then derive the limit $\mathbf{P}(X > 0) \to 1$ from (8), by Chebyshev's inequality.

Here we give the proof of the first inequality of (8). The proof of the second bound of (8) is similar. Write $\mathbf{E}X = na$, where $a = \mathbf{E}\mathbb{I}_v$. Denote $\varkappa = d^2nm^{-1}$. Let us show that uniformly in $n, m \to \infty$ satisfying (4)

$$a = (1 + \varkappa + r)e^{-\varkappa + r}. \tag{9}$$

Here r is the remainder term of order $r = O(m^{-1} + nm^{-2})$. We have

$$a = \mathbf{P}\bigl(d(v) = 0\bigr) + \mathbf{P}\bigl(d(v) = 1\bigr) = \alpha^{n-1} + (n-1)(1-\alpha)\alpha^{n-2},$$

where $\alpha = \binom{m-d}{d}\binom{m}{d}^{-1}$ is the probability that two given vertices are non-adjacent. Invoking the expansion $\alpha = 1 - d^2m^{-1} + O(m^{-2})$, see (7), we obtain (9)

$$a = \bigl(n(1-\alpha) + 2\alpha - 1\bigr)e^{(n-2)\ln\alpha} = (1 + \varkappa + r)e^{-\varkappa + r}. \tag{10}$$

Now a simple caclulation shows that $\mathbf{E}X = na \to +\infty$ for n, m satisfying (4). $\square$

Lemma 3. *Let $s_1, s_2, \ldots, s_n$ be independent random subsets of $W = \{w_1, \ldots, w_m\}$ such that, for every i, s_i is uniformly distributed in the class of subsets of W of size 2. Assume that*

$$n = 2^{-1}m\big(\ln m + \ln\ln m + \omega(m)\big) \tag{11}$$

for some $\omega(m) \to +\infty$ satisfying $\omega(m) = o(\ln\ln m)$ as $m \to \infty$. Then for any $c > 0$, the number n' of distinct subsets among $s_1, \ldots, s_n$ satisfies as $m \to \infty$

$$\mathbf{P}\Big(n' \geqslant 2^{-1}m\big(\ln m + \ln\ln m + \omega(m) - c\big)\Big) = 1 - o(1) \tag{12}$$

Proof. Denote $A(c) = 2^{-1}m\big(\ln m + \ln\ln m + \omega(m) - c\big)$ and write $A =: A(0)$, $A' := A(c)$.

Assume that sets $s_1, s_2, \ldots$ are drawn independently at random one after another and N_i counts the number of distinct sets among $s_1, \ldots, s_i$. As long as we have $N_{i-1} < A'$ we write $\mathbb{I}_i = 1$ in the case where s_i differs from all previously drawn sets $s_1, \ldots, s_{i-1}$, and write $\mathbb{I}_i = 0$ otherwise. After the number of distinct sets reaches A' (i.e., for i satisfying $N_i \geqslant A'$) we put $\mathbb{I}_i = 1$ in either case. Denote $X = \mathbb{I}_1 + \cdots + \mathbb{I}_n$. We have $n' \geqslant A'$ whenever $X \geqslant A'$. Note that for every i, we have

$$\mathbf{P}(\mathbb{I}_i = 1) \geqslant 1 - \frac{A' - 1}{\binom{m}{2}} =: p'.$$

A coupling with a binomial random variable $Y \sim Bin(n, p')$ gives $\mathbf{P}(X \leqslant A') \leqslant \mathbf{P}(Y \leqslant A')$. Next we apply Chernoff's inequality to the latter probability,

$$\mathbf{P}(Y \leqslant A') \leqslant \exp\big\{-2^{-1}(np' - A')^2/(np')\big\}.$$

A simple calculation shows, that $(np' - A')^2/(np') \to +\infty$ as $m \to \infty$. Therefore, we obtain

$$\mathbf{P}(n' \geqslant A') = \mathbf{P}(X \geqslant A') \geqslant \mathbf{P}(Y \geqslant A') = 1 - o(1)$$

thus completing the proof.

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REZIUMĖ

Apie tolygių atsitiktinių sankirtų grafų hamiltoniškumą

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Darbe nagrinėjamas Hamiltono ciklo egzistavimas tolygiame atsitiktiniame sankirtų grafe $G_{n,m,d}$. Tai grafas, turintis n viršnių. Kiekviena viršnė iš duotos m raktų aibės atsitiktinai ir nepriklausomai išsirenka d raktų rinkinį. Dvi viršnės jungiamos briauna, jei jos turi bent vieną bendrą raktą. Darbe parodoma, jog su tikimybe, artėjančia prie 1, grafas $G_{n,m,d}$ turi Hamiltono ciklą, jeigu $n = 2^{-1}m(\ln m + \ln\ln m + \omega(m))$, kur $\omega(m) \to +\infty$, kai $m \to \infty$.

Raktiniai žodžiai: atsitiktinis grafas, sankirtų grafas, Hamiltono ciklas, klasterizavimas.