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Left permutable multiplicative sets and left strongly prime ideals in rings

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Abstract. Left permutable multiplicative sets $\mathcal{S}$ for an associative ring R are defined. Particularly, this notion includes commutative multiplicative sets of the associative ring. We also define the notion of the left $\mathcal{S}$-ideal and prove, that each left $\mathcal{S}$-ideal, maximal with respect to being disjoint from $\mathcal{S}$, is left strongly prime.

Keywords: (left) strongly prime ideal; insulator; multiplicative set; left strongly prime radical.

All considered rings are associative with identity element. $A \subset B$ means that A is a proper subset of B.

Let R be a nonzero ring. We recall that a left ideal $\mathfrak{p} \subset R$ is called *(left) strongly prime* if for each $u \notin \mathfrak{p}$ there exists a finite set $\alpha_1, ..., \alpha_n \in R, \quad n = n(u)$, such that $r\alpha_1 u, ..., r\alpha_n u \in \mathfrak{p}$, where $r \in R$, implies that $r \in \mathfrak{p}$. Each such subset $\{\alpha_1 u, ..., \alpha_n u\}$ is called *an insulator* of u for $\mathfrak{p}$. Evidently, maximal left ideals are strongly prime. When R is commutative, strongly prime ideals are precisely prime ideals. When the zero ideal is left strongly prime, we obtain the notion of a left strongly prime ring, introduced and investigated in [2]. Basic properties of the left strongly prime ideals are considered in [3, 4, 7].

1. Multiplicative sets and left strongly prime ideals

Recall that a subset $\mathcal{S}$ of a ring R is multiplicative, if it is multiplicatively closed and contains the identity element of R. By well known important results in commutative algebra, each ideal $\mathfrak{p} \subset R$, maximal with respect to being disjoint from $\mathcal{S}$, i.e., $\mathfrak{p} \cap \mathcal{S} = \emptyset$, is prime. Also, the set $R \backslash \mathfrak{p}$ is multiplicative. There are some generalizations of this result for noncommutative rings. Recall that a subset $\mathcal{S} \subseteq R$ of an associative ring is an *m-system* if $1_R \in \mathcal{S}$ and for each $a, b \in \mathcal{S}$, there exists $r \in R$ such that $arb \in \mathcal{S}$. Main properties of the m-systems are also well known: a complement of a prime (two-sided) ideal is an m-system, and each two-sided ideal maximal with respect to being disjoint from S is prime. This result was generalized for two-sided strongly prime ideals in R, when R is viewed as the left module over its multiplication ring (see [5], Proposition 3.4 and Theorem 3.5). All mentioned results hold only for the two-sided ideals. We note that no methods, allowing to construct left strongly prime ideals using multiplicative sets of a noncommutative ring, were known. We introduce the notion of a left permutable multiplicative set for an associative ring and get their relations with a left strongly prime ideals of a ring.

Let R be a nonzero ring. Denote by $\mathfrak{F}(R)$ the set which elements are all finite subsets of the ring R, endowed with the structure of monoid under standard multiplication, induced by the multiplication of the ring: if $\{a_1,...,a_n\} = \mathfrak{a},\ \{b_1,...,b_m\} = \mathfrak{b} \in \mathfrak{F}(R)$, then $\mathfrak{a}\mathfrak{b} = \{a_i b_j\},\ \ 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m$.

In what follows $\mathcal{S}$ will be a submonoid of $\mathfrak{F}(R)$. This means that elements of $\mathcal{S}$ are finite subsets of the ring R, $\{1_R\} \in \mathcal{S}$ and $\mathcal{S}$ is closed under multiplication mentioned above. Sending $r \in R$ to the singleton $\{r\} \in \mathfrak{F}(R)$ we obtain a canonical inclusion of the multiplicative monoid R into $\mathfrak{F}(R)$.

We call a submonoid $\mathcal{S} \subseteq \mathfrak{F}(R)$ a *left permutable multiplicative set*, if for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{S}$, there exist $\mathfrak{c}, \mathfrak{d} \in \mathcal{S}$, depending from $\mathfrak{a}$ and $\mathfrak{b}$, such that $\mathfrak{a}\mathfrak{b}\mathfrak{c} = \mathfrak{b}\mathfrak{d}$.

Particularly, such are commutative multiplicative subsets of the ring R. Let $\mathfrak{s} = \{a_1,...,a_m\} \subseteq R$ be any finite subset. Then $\mathcal{S} = \{\mathfrak{s}^m, m \geqslant 0\}$ is commutative, so, left permutable multiplicative set in $\mathfrak{F}(R)$.

Note that in terms of elements the left permutability for $\mathcal{S} \subseteq \mathfrak{F}(R)$ particularly means that for all $a_i \in \mathfrak{a}, b_j \in \mathfrak{b}, c_k \in \mathfrak{c}$, there exist elements $b_m \in \mathfrak{b},\ d_n \in \mathfrak{d}$, such that $a_i b_j c_k = b_m d_n$.

In the sequel we will need the following lemma.

LEMMA 1. *Let* $\mathfrak{a}_1,...,\mathfrak{a}_n, \mathfrak{b}_1,...,\mathfrak{b}_n \in \mathcal{S}$, *where* $\mathcal{S}$ *is a left permutable multiplicative set. Then there exist* $\mathfrak{c}_n, \mathfrak{d}_1,...,\mathfrak{d}_n \in \mathcal{S}$ *, having property* $\mathfrak{a}_k\mathfrak{b}_k\mathfrak{c}_n = \mathfrak{b}_k\mathfrak{d}_k$, *for all* $1 \leqslant k \leqslant n$.

Proof. We use induction. The lemma is true for $n = 1$ by the definition of a left permutable multiplicative set. Take $\mathfrak{a}_{n+1}, \mathfrak{b}_{n+1} \in \mathcal{S}$. Let $\mathfrak{c}_n, \mathfrak{d}_1,...,\mathfrak{d}_n \in \mathcal{S}$ be such, that $\mathfrak{a}_k\mathfrak{b}_k\mathfrak{c}_n = \mathfrak{b}_k\mathfrak{d}_k$ for $1 \leqslant k \leqslant n$. Take $\mathfrak{c}, \mathfrak{d} \in \mathcal{S}$ for which $\mathfrak{a}_{n+1}\mathfrak{b}_{n+1}\mathfrak{c} = \mathfrak{b}_{n+1}\mathfrak{d}$ and $\mathfrak{e}, \mathfrak{f} \in \mathcal{S}$ for which $\mathfrak{c}_n\mathfrak{c}\mathfrak{e} = \mathfrak{c}\mathfrak{f}$. Then for $\mathfrak{c}_{n+1} = \mathfrak{c}_n\mathfrak{c}\mathfrak{e} = \mathfrak{c}\mathfrak{f}$ our assertion holds for all $1 \leqslant k \leqslant n+1$.

We say that a left ideal $L \subset R$ is *disjoint from* $\mathcal{S} \subset \mathfrak{F}(R)$ if $\mathfrak{a} \nsubseteq L$ for all $\mathfrak{a} \in \mathcal{S}$. By this definition, L can contain some, but not all elements of $\mathfrak{a}$. Evidently, a left ideal disjoint from $\mathcal{S}$, exists if and only if the zero subset $\{0\} \notin \mathcal{S}$.

Now we introduce the main notion, which enables us to get the main results of this paper.

A left ideal $L \subseteq R$ is called *an* $\mathcal{S}$*-ideal*, if $L\mathfrak{a} \subseteq L$ for all $\mathfrak{a} \in \mathcal{S} \subseteq \mathfrak{F}(R)$.

This means, that $La \subseteq L$ for all elements $a \in \mathfrak{a}$ where $\mathfrak{a} \in \mathcal{S}$. Evidently, two-sided ideals are $\mathcal{S}$-ideals.

THEOREM 1. *Let* $\mathcal{S} \subset \mathfrak{F}(R)$ *be a left permutable multiplicative set. Then each left* $\mathcal{S}$*-ideal* $\mathfrak{p} \subset R$, *maximal with respect to being disjoint from* $\mathcal{S}$, *is left strongly prime. Such a left* $\mathcal{S}$*-ideal exists if and only if* $\{0\} \notin \mathcal{S}$

Proof. Consider the family $\mathfrak{L} = \{L_i, i \in I\}$ of all proper left $\mathcal{S}$-ideals of a ring R, disjoint from $\mathcal{S}$. As noted above, $\mathfrak{L}$ is not empty if and only if $\{0\} \notin \mathcal{S}$, because then $L = 0$ belongs to the family. So, when $\{0\} \notin \mathcal{S}$, then $\mathfrak{L}$ is not empty and inductive, because elements of $\mathcal{S}$ contain only finite number of elements of the ring. Thus, by

Zorn's lemma, $\mathfrak{L}$ contains at least one maximal element $\mathfrak{p} \in \mathfrak{L}$. We show that $\mathfrak{p}$ is left strongly prime ideal. Let $u \notin \mathfrak{p}$. Denote $u\mathcal{S} = \{ua\} \subseteq R$ where $a \in \mathfrak{a} \in \mathcal{S}$.

Then $\mathfrak{p} + Ru\mathcal{S}$ is an $\mathcal{S}$-ideal properly containing $\mathfrak{p}$, so $\mathfrak{a} \subseteq \mathfrak{p} + Ru\mathcal{S}$ for some $\mathfrak{a} \in \mathcal{S}$. Thus, for each $a_k \in \mathfrak{a}$ we have expressions

$$a_k = p_k + \alpha_{k1} u f_{k1} + \ldots + \alpha_{km} u f_{km},$$

where $p_k \in \mathfrak{p}$, $\alpha_{ki} \in R$, $f_{ki} \in \mathfrak{f}_{ki} \in \mathcal{S}$. We show that the finite set $\{\alpha_{ki} u\}$, consisting of all elements $\alpha_{ki} u$ in these expressions, is an insulator of u for $\mathfrak{p}$. Indeed, let all elements $r\alpha_{ki} u \in \mathfrak{p}$ for some $r \in R$. Then, because $\mathfrak{p}$ is an $\mathcal{S}$-ideal, elements $r\alpha_{ki} u f_{ki}$ also belong to $\mathfrak{p}$. Thus all $ra_k \in \mathfrak{p}$, so $r\mathfrak{a} \subseteq \mathfrak{p}$. If $r \notin \mathfrak{p}$, then, analogously, the left $\mathcal{S}$-ideal $\mathfrak{p} + Rr\mathcal{S}$ contains some subset $\mathfrak{b} \in \mathcal{S}$. So, for all $b_l \in \mathfrak{b}$ we would have

$$b_l = q_l + \beta_{l1} r g_{l1} + \ldots + \beta_{ln} r g_{ln},$$

with some $q_l \in \mathfrak{p}$, $\beta_{lj} \in R$, $g_{lj} \in \mathfrak{g}_{lj} \in \mathcal{S}$. By Lemma 1, there exist $\mathfrak{c}, \mathfrak{d}_{lj} \in \mathcal{S}$, such that $\mathfrak{g}_{lj}\mathfrak{a}\mathfrak{c} = \mathfrak{a}\mathfrak{d}_{lj}$ for all $\mathfrak{g}_{lj}$. Multiplying all expressions for b_l by $\mathfrak{a}\mathfrak{c}$, we immediately obtain that $\mathfrak{b}\mathfrak{a}\mathfrak{c} \subseteq \mathfrak{p}$. But $\mathfrak{b}\mathfrak{a}\mathfrak{c} \in \mathcal{S}$, so this is a contradiction with $\mathfrak{p}$ being disjoint from $\mathcal{S}$. So $r \in \mathfrak{p}$ and $\mathfrak{p}$ is left strongly prime.

Moreover, we have showed that elements from $\mathcal{S}$ are insulators of the 1_R for $\mathfrak{p}$, i.e., $r\mathfrak{a} \subseteq \mathfrak{p}$ implies $r \in \mathfrak{p}$ for all $\mathfrak{a} \in \mathcal{S}$.

Remark 1. Evidently, if each left $\mathcal{S}$-ideal $L_\alpha \in \mathfrak{L}$ contains some left ideal L, then the same argument gives us a left strongly prime $\mathcal{S}$-ideal $\mathfrak{p}$ containing L.

COROLLARY 1. *Let $\mathfrak{s} = \{a_1, \ldots, a_n\} \subseteq R$ be a non-nilpotent subset, $\mathcal{S} = \{\mathfrak{s}^m, m \geqslant 0\}$. Then there exists a left strongly prime $\mathcal{S}$-ideal disjoint from $\mathcal{S}$.*

Proof. Indeed, $\mathcal{S}$ is commutative and, because $\mathfrak{s}$ is non-nilpotent subset of R, $\{0\} \notin \mathcal{S}$, so the zero ideal of R is disjoint from $\mathcal{S}$ and $\mathfrak{L} \neq \emptyset$. So, by Theorem 1, there exists left strongly prime ideal $\mathfrak{p}$, having properties $\mathfrak{p}\mathfrak{s} \subseteq \mathfrak{p}$ and $\mathfrak{s}^m \not\subseteq \mathfrak{p}$ for all $m \in \mathbf{N}$. Particularly $\mathfrak{s} \not\subseteq \mathfrak{p}$.

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REZIUMĖ

A. Kaučikas. Kairiosios perstatomos multiplikatyviosios aibės ir stipriai pirminiai žiedų idealai

Apibrėžtos kairiosios perstatomos multiplikatyviosios žiedų aibės $\mathcal{S}$ ir kairieji $\mathcal{S}$-idealai. Įrodyta, kad kiekvienas maksimalus kairysis žiedo idealas, nesikertantis su $\mathcal{S}$, yra stipriai pirminis.

Raktiniai žodžiai: (kairysis) stipriai pirminis idealas, izoliatorius, multiplikatyvioji aibė; kairysis stipriai pirminis radikalas.