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Poisson-type approximation for sums of 1-dependent indicators

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Abstract. The sum of 1-dependent indicators is approximated by two-parametric Poisson type signed measure. Estimates are obtained for the local and total variation norms.

Keywords: m-dependent random variables, signed compound Poisson measure, total variation norm, local norm.

Let $X_j,\ j=1,2,\dots,n$ be a triangular array of 1-dependent Bernoulli variables, $P(X_j=1)=p_j=1-P(X_j=0)$. We denote the distribution and characteristic function of $S_n=X_j+X_{j+1}+\dots+X_{j+n-1}$ by F_n and $\widehat{F}_n(t)$, respectively. Let

$$\lambda_m=\sum_{k=1}^{n}p_k^m,\quad m=1,2,$$

$$a_{i,j}=\widehat{\mathrm{E}}X_jX_{j+1}\dots X_{j+i-1}$$

$$:=\sum_{l=1}^{i}(-1)^{l-1}\sum_{\substack{i_1+\dots+i_l=i\\ i_m\geqslant 1}}EX_j\dots X_{j+i_1-1}E_{j+i_1}\dots X_{j+i_1+i_2-1}\dots EX_{j+i_1+\dots+i_{l-1}}\dots X_{j+i-1},$$

$$R_1=\sum_{k=1}^{n}(p_k^3+|a_{3,k}|+|a_{2,k}|p_k),\qquad R_2=\sum_{k=1}^{n}(|a_{4,k}|+a_{2,k}^2+|a_{3,k}|p_k),$$

$$R_3=\sum_{k=1}^{n}(|a_{5,k}|+|a_{2,k}a_{3,k}|).$$

For approximation of F_n we use a signed compound Poisson measure D with the Fourier–Stieltjes transform

$$\hat{D}(t)=\exp\Bigl\{\lambda_1(\mathrm{e}^{\mathrm{i}t}-1)+\Bigl(\sum_{k=2}^{n}a_{2,k}-\frac{1}{2}\lambda_2\Bigr)(\mathrm{e}^{\mathrm{i}t}-1)^2\Bigr\}.$$

Let $W=F_n-D$. The total variation norm and the local norm of W are denoted by $\|W\|=\sum_{k=-\infty}^{\infty}|W\{k\}|$, $\|W\|_\infty=\sup_{k\in\mathbb{Z}}|W\{k\}|$, respectively.

The normal approximation for m-dependent variables was investigated in numerous papers, see [4] and [6] and references therein. Signed compound Poisson approximations have advantages over the more traditional normal or Poisson approximations, since allow for the usage of stronger metrics and, as a rule, provide sharper estimates. A classical example is the result of Kruopis [5], who proved that, in the case of independent indicators, D is uniformly sharper than both, the normal and Poisson, approximations. In the case of independent summands, approximations by signed compound Poisson measures are widely used, see, for example, [1–3] and references therein. On the other hand, there are just a few results for the sums of dependent random variables. Our purpose is to obtain an analogue of Kruopis [5] result for 1-dependent random variables. We assume that all p_j are small and the dependence of variables is weak. More precisely, let

$$\max_{1\leqslant j\leqslant n} p_j = \mathrm{o}(1), \qquad \sum_{j=1}^{n} a_{2,j} = \mathrm{o}(\lambda_1), \qquad n \to \infty. \tag{1}$$

THEOREM 1. *Let conditions* (1) *hold. Then, for all* $n = 1, 2, \ldots,$

$$\|W\|_\infty = \mathrm{O}\left\{R_1 \min\left(1, \frac{1}{\lambda_1^2}\right) + R_2 \min\left(1, \frac{1}{\lambda_1^{5/2}}\right) + R_3 \min\left(1, \frac{1}{\lambda_1^3}\right)\right\},$$

$$\|W\| = \mathrm{O}\left\{R_1 \min\left(1, \frac{1}{\lambda_1^{3/2}}\right) + R_2 \min\left(1, \frac{1}{\lambda_1^2}\right) + R_3 \min\left(1, \frac{1}{\lambda_1^{5/2}}\right)\right\}.$$

Remark. In principle, the first condition in (1) can be replaced by the weaker one, requiring max p_j to be smaller than some absolute constant. Unfortunately, due to the estimates used in the proofs, the constant is very small.

Example 1. It is easy to check that if, in Theorem 1, $X_1, X_2, \ldots, X_n$ are independent r.v., then

$$\|W\|_\infty = \mathrm{O}\left(\sum_{k=1}^{n} p_k^3 \min\left(1, \frac{1}{(\lambda_1)^2}\right)\right), \qquad \|W\| = \mathrm{O}\left(\sum_{k=1}^{n} p_k^3 \min\left(1, \frac{1}{(\lambda_1)^{3/2}}\right)\right),$$

which have the same order of accuracy as similar results from [5].

Example 2. Let us consider the case of 2-way runs, that is let ξ_k be independent Bernoulli variables with $P(\xi_k = 1) = 1 - P(\xi_k = 0) = \alpha$ and let $X_1 = \xi_1\xi_2$, $X_2 = \xi_2\xi_3$, $X_3 = \xi_3\xi_4, \ldots$. If $\alpha \to 0$, then

$$\|W\|_\infty = \mathrm{O}\left(\frac{1}{n}\right), \qquad \|W\| = \mathrm{O}\left(\frac{\alpha}{\sqrt{n}}\right).$$

It seems that the local estimate was not previously considered. The second estimate has the same order of accuracy as given in Theorem 5.2 from [1].

For the proof of Theorem we need auxiliary results. For the sake of brevity set $z = \mathrm{i}t$, $x = \mathrm{e}^{\mathrm{i}t} - 1$. Moreover, we denote by θ all quantities satisfying $|\theta| \leqslant 1$ and use C for all absolute constants, which may vary from line to line. Let $\varphi_1 = \varphi_1(z) = \mathrm{E}\,\mathrm{e}^{zX_1}$,

$$\varphi_k = \frac{\mathrm{E}\,\mathrm{e}^{zS_k}}{\mathrm{E}\,\mathrm{e}^{zS_{k-1}}}, \quad w_n(\mathrm{i}t) := \sqrt{\mathrm{E}\,|\mathrm{e}^{\mathrm{i}tX_j} - 1|^2}, \quad k = 2, 3, \dots, n.$$

The following lemma follows from Lemma 3.1 and Lemma 3.2 in [4].

LEMMA 1. *Let* (1) *hold. Then, for* $k = 1, 2, \dots n$, *and all real* t

$$\varphi_k = \mathrm{E}\,\mathrm{e}^{zX_k} + \sum_{j=1}^{k-1} \frac{\widehat{\mathrm{E}}\,(\mathrm{e}^{zX_j} - 1)(\mathrm{e}^{zX_{j+1}} - 1)\dots(\mathrm{e}^{zX_k} - 1)}{\varphi_j \varphi_{j+1} \cdots \varphi_{k-1}},$$

$$\begin{aligned} |\varphi_k - 1| &\leqslant |\mathrm{E}\,\mathrm{e}^{zX_k} - 1| + 2\sqrt{\mathrm{E}\,|\mathrm{e}^{zX_{k-1}} - 1|^2 \mathrm{E}\,|\mathrm{e}^{zX_k} - 1|^2}/(1 - 4w_n(z)) \\ &\leqslant p_k|\mathrm{e}^z - 1| + 12\sqrt{p_k p_{k-1}}|\mathrm{e}^z - 1| \leqslant C(p_k + p_{k-1})|\mathrm{e}^z - 1|, \end{aligned}$$

$$|\ln \widehat{F}_n(t) - \lambda_1(\mathrm{e}^z - 1)| \leqslant \lambda_1\Big(\sum_{k=1}^{n} |a_{2,k}|/\lambda_1 + 90 \max_{1 \leqslant k \leqslant n} \sqrt{p_k}\Big)|\mathrm{e}^z - 1|^2.$$

LEMMA 2 ([4]). *Let* $Y_1, Y_2, \dots Y_k$ *be 1-dependent random variables with* $\mathrm{E}|Y_j|^2 < \infty$, $j = 1, \dots, k$. *Then*

$$\big|\widehat{\mathrm{E}}\,Y_1 Y_2 \dots Y_j\big| \leqslant 2^{j-1} \prod_{k=1}^{j} \sqrt{\mathrm{E}\,|Y_k|^2}.$$

LEMMA 3. *Let* (1) *be satisfied. Then, for all* $|t| \leqslant \pi$,

$$\max\big\{|\widehat{F}_n(t)|, |\widehat{D}(t)|\big\} \leqslant \exp\{-C\lambda_1 t^2\}.$$

Proof. Note that

$$\big|\widehat{F}_n(t)\big| \leqslant \big|\exp\big\{\lambda_1(\mathrm{e}^z - 1)\big\}\big| \exp\Big\{\big|\ln \widehat{F}_n(t) - \lambda_1(\mathrm{e}^z - 1)\big|\Big\}$$

and apply Lemma 1. The estimate for $\widehat{D}(t)$ follows directly from its definition and (1).

LEMMA 4. *Let condition* (1) *be satisfied. Then, for* $k \geqslant 5$, *we have*

$$\begin{aligned} \varphi_k - 1 = {} & p_k x + a_{2,k} x^2 + (a_{3,k} - a_{2,k} p_{k-1}) x^3 \\ & + \big(a_{4,k} - a_{3,k}(p_{k-2} + p_{k-1}) - a_{2,k} a_{2,k-1}\big) x^4 \\ & + \big(a_{5,k} - a_{3,k}(a_{2,k-1} + a_{2,k-2}) - a_{2,k} a_{3,k-1}\big) x^5 \\ & + C\theta\big(p_k^3 + p_{k-1}^3 + p_{k-2}^3 + p_{k-3}^3 + p_{k-4}^3 + p_{k-5}^3\big)|x|^3. \qquad (2) \end{aligned}$$

Moreover, for $k = 2, 3, 4$ *the estimate* (2) *holds with* $a_{3,k} = a_{4,k} = a_{5,k} = 0$, $a_{4,k} = a_{5,k} = 0$, $a_{5,k} = 0$, *respectively.*

Proof. Applying Lemma 1 we obtain

$$\varphi_k - 1 = p_k x + \frac{a_{2,k}x^2}{\varphi_{k-1}} + \frac{a_{3,k}x^3}{\varphi_{k-2}\varphi_{k-1}} + \frac{a_{4,k}x^4}{\varphi_{k-3}\varphi_{k-2}\varphi_{k-1}} + \frac{a_{5,k}x^5}{\varphi_{k-4}\varphi_{k-3}\varphi_{k-2}\varphi_{k-1}}$$
$$+ \sum_{j=1}^{k-5} \frac{\widehat{\mathrm{E}}\,(\mathrm{e}^{zX_j} - 1)(\mathrm{e}^{zX_{j+1}} - 1)\dots(\mathrm{e}^{zX_k} - 1)}{\varphi_j\varphi_{j+1}\cdots\varphi_{k-1}} \tag{3}$$

and

$$\frac{1}{|\varphi_k|} \leqslant \frac{1}{1 - |1 - \varphi_k|} \leqslant \frac{5}{4}.$$

From Lemma 2 we obtain

$$\left|\widehat{\mathrm{E}}\,(\mathrm{e}^{zX_j} - 1)(\mathrm{e}^{zX_{j+1}} - 1)\dots(\mathrm{e}^{zX_k} - 1)\right|$$
$$\leqslant 2^{k-j} \prod_{m=j}^{k} \sqrt{\mathrm{E}\,|\mathrm{e}^{zX_m} - 1|^2} = 2^{k-j}|x|^{k-j+1}\sqrt{p_j p_{j+1}\cdots p_k}.$$

Noting that, due to (1), we can assume p to be small. Therefore,

$$\left|\sum_{j=1}^{k-5} \frac{\widehat{\mathrm{E}}\,(\mathrm{e}^{zX_j} - 1)(\mathrm{e}^{zX_{j+1}} - 1)\dots(\mathrm{e}^{zX_k} - 1)}{\varphi_j\varphi_{j+1}\cdots\varphi_{k-1}}\right|$$
$$\leqslant \sqrt{p_{k-5}\dots p_k}|x|^3 \sum_{j=1}^{k-5} 5^{k-j}\left(\frac{1}{180}\right)^{k-j-6+1}$$
$$\leqslant C\sqrt{p_{k-5}\dots p_k}|x|^3 \sum_{j=1}^{k-5}\left(\frac{5}{180}\right)^{k-j} \leqslant C\sqrt{p_{k-5}\dots p_k}|x|^3. \tag{4}$$

From (3) and (4) we get

$$\frac{1}{\varphi_k} = 1 + (1 - \varphi_k) + (1 - \varphi_k)^2 + C\theta(p_k^2 + p_{k-1}^2)|x|^3$$
$$= 1 - p_k x - \frac{a_{2,k}x^2}{\varphi_{k-1}} - \frac{a_{3,k}x^3}{\varphi_{k-1}\varphi_{k-2}} + (1 - \varphi_k)^2 + C\theta(p_k^2 + p_{k-1}^2)|x|^3$$
$$= 1 - p_k x - a_{2,k}x^2 - a_{3,k}x^3 + C\theta(p_k^2 + p_{k-1}^2 + p_{k-2}^2 + p_{k-3}^2)|x|^3.$$

Putting the last expression into (3) we complete the proof of Lemma 4, for $k \geqslant 5$. The cases $k = 2, 3, 4$ are proved similarly.

LEMMA 5. *Let condition* (1) *be satisfied. Then, for* $k \geqslant 5$,

$$\begin{aligned}
\ln\varphi_k &= p_k x + \Big(a_{2,k} - \frac{1}{2}p_k^2\Big)x^2 + \big(a_{3,k} - a_{2,k}(p_k + p_{k-1})\big)x^3 \\
&\quad + \bigg(a_{4,k} - a_{3,k}(p_k + p_{k-1} + p_{k-2}) - a_{2,k}\Big(\frac{1}{2}a_{2,k} + a_{2,k-1}\Big)\bigg)x^4 \\
&\quad + \big(a_{5,k} - a_{3,k}(a_{2,k} + a_{2,k-1} + a_{2,k-2}) - a_{2,k}a_{3,k-1})\big)x^5 \\
&\quad + C\theta\big(p_k^3 + p_{k-1}^3 + p_{k-2}^3 + p_{k-3}^3 + p_{k-4}^3 + p_{k-5}^3\big)|x|^3, \\
\frac{\mathrm{d}}{\mathrm{d}t}\,\varphi_k &= \mathrm{i}\mathrm{e}^z\Big[p_k + 2a_{2,k}x + 3(a_{3,k} - a_{2,k}p_{k-1})x^2 \\
&\quad + 4\big(a_{4,k} - a_{3,k}(p_{k-2} + p_{k-1}) - a_{2,k}a_{2,k-1}\big)x^3 \\
&\quad + 5\big(a_{5,k} - a_{3,k}(a_{2,k-1} + a_{2,k-2}) - a_{2,k}a_{3,k-1}\big)x^4\Big] \\
&\quad + C\theta\big(p_k^3 + p_{k-1}^3 + p_{k-2}^3 + p_{k-3}^3 + p_{k-4}^3 + p_{k-5}^3\big)|x|^2, \\
\frac{\mathrm{d}}{\mathrm{d}t}\ln\varphi_k &= \mathrm{i}\mathrm{e}^z\Big[p_k + \big(2a_{2,k} - p_k^2\big)x + 3\big(a_{3,k} - a_{2,k}(p_k + p_{k-1})\big)x^2 \\
&\quad + 4\bigg(a_{4,k} - a_{3,k}(p_k + p_{k-1} + p_{k-2}) - a_{2,k}\Big(\frac{1}{2}a_{2,k} + a_{2,k-1}\Big)\bigg)x^3 \\
&\quad + 5\big(a_{5,k} - a_{3,k}(a_{2,k} + a_{2,k-1} + a_{2,k-2}) - a_{2,k}a_{3,k-1}\big)x^4\Big] \\
&\quad + C\theta\big(p_k^3 + p_{k-1}^3 + p_{k-2}^3 + p_{k-3}^3 + p_{k-4}^3 + p_{k-5}^3\big)|x|^2.
\end{aligned}$$

Moreover, for $k = 2, 3, 4$ *the estimate* (2) *holds with* $a_{3,k} = a_{4,k} = a_{5,k} = 0$, $a_{4,k} = a_{5,k} = 0$, $a_{5,k} = 0$, *respectively.*

Proof. Proof of Lemma 5, in principle, repeats the proof of Lemma 4 and, therefore, is omitted.

LEMMA 6. *Let* (1) *be satisfied and let* $|t| \leqslant \pi$. *Then*

$$\begin{aligned}
&\big|\widehat{F}_n(t) - \widehat{D}(t)\big| \leqslant C\exp\big\{-C\lambda_1 t^2\big\}\big[R_1|t|^3 + R_2|t|^4 + R_3|t|^5\big], \\
&\big|\big(\widehat{F}_n(t)\mathrm{e}^{-\mathrm{i}t\lambda_1} - \widehat{D}(t)\mathrm{e}^{-\mathrm{i}t\lambda_1}\big)'\big| \leqslant C\exp\big\{-C\lambda_1 t^2\big\}\big[R_1|t|^2 + R_2|t|^3 + R_3|t|^4\big].
\end{aligned}$$

Proof. Applying Lemma 3 we get

$$\big|\widehat{F}_n(t) - \widehat{D}(t)\big| \leqslant C\exp\big\{-C\lambda_1 t^2\big\}\big|\ln\widehat{F}_n(t) - \ln\widehat{D}(t)\big|.$$

Note that $\ln\widehat{F}_n(t) = \sum_{k=1}^n \ln\varphi_k$. Consequently, from Lemma 5 we get the first estimate. The second estimate is proved similarly.

Proof of Theorem 1. The local estimate follows directly from Lemma 6 and formula of inversion:

$$\|W\|_\infty \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widehat{W}(t)|\,\mathrm{d}t.$$

Let $\beta = \max(1, \sqrt{\lambda_1})$. The proof for the total variation norm follows from Lemma 6 and the well-known estimate

$$\|W\|^2 \leqslant C\int_{-\pi}^{\pi}\left(\beta|\widehat{W}(t)|^2 + \frac{1}{\beta}\big|\big(\mathrm{e}^{-\mathrm{i}t\lambda_1}\widehat{W}(t)\big)'\big|^2\right)\mathrm{d}t.$$

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REZIUMĖ

J. Kelmelytė, V. Čekanavičius. 1-priklausomų indikatorių sumos Puasono tipo aproksimacija

1-priklausomų indikatorių suma aproksimuojama dviparametriniu Puasono tipo ženklą keičiančiu matu. Gauti lokalūs ir pilnosios variacijos aproksimacijos tikslumo įverčiai.

Raktiniai žodžiai: m-priklausomi atsitiktiniai dydžiai, ženklą keičiantis sudėtinis Puasono matas, pilnosios variacijos norma, lokali norma.