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$\boldsymbol{p}$-variation of Ornstein–Uhlenbeck type processes*

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Abstract. The p-variation of sample paths of Ornstein–Uhlenbeck type processes is investigated. It is shown that the p-variation index of such a process is the same as the p-variation index of the driving Lévy process, provided this process is of unbounded total variation.

Keywords: Ornstein–Uhlenbeck type process, Lévy process, p-variation.

1. Introduction

Assume that $X = \{X_t,\ t \geqslant 0\}$ is a real valued Lévy process, i.e., a process with stationary and independent increments, which starts at the origin at $t = 0$ and has almost all cádlág trajectories. Fix a constant $\lambda > 0$ and consider the following stochastic differential equation

$$\mathrm{d}Y_u = -\lambda Y_u\,\mathrm{d}u + \mathrm{d}X_u, \quad u \geqslant 0, \tag{1}$$

which can be understood in the sense of semimartingales (see [6]), since such are Lévy processes. Multiplying both sides of (1) by $\mathrm{e}^{\lambda u}$ and integrating from 0 to t, we easily obtain the solution as

$$Y_t = \mathrm{e}^{-\lambda t} Y_0 + \mathrm{e}^{-\lambda t} \int_0^t \mathrm{e}^{\lambda u}\,\mathrm{d}X_u, \tag{2}$$

which is called an Ornstein–Uhlenbeck type (OU-type) process, generated by X (see [7, §17], [2, §15.3]). The stochastic integral in (2) can also be understood *pathwise* (see [1, Thm. 7.14 and Prop. 3.9.1]) in the refinement-Riemann–Stieltjes sense (see [4, p. 2]) and thus allows applications of several results related to the path properties of Y_t. In fact, we will show that the p-variation properties of Y_t are the same as for X_t, provided X_t is of unbounded total variation.

2. Preliminaries and results

Throughout we will fix a $T > 0$ and will omit it in the notations, since the value of T will be unimportant. We will say that a process $X_s, 0 \leqslant s \leqslant T$ (or, in particular, a function $f\colon [0,T] \to \mathbb{R}$) has a finite p-variation index $\upsilon(X)$ almost surely if

$$\upsilon(X) = \inf\bigl\{p > 0\colon\ \upsilon_p(X) < \infty \quad \text{almost surely}\bigr\} \tag{3}$$

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is finite, where

$$v_p(X) = \sup\left\{ \sum_{k=0}^{m-1} |X_{t_{k+1}} - X_{t_k}|^p \colon 0 = t_0 < t_1 < \ldots < t_m = T, m = 1, 2, \ldots \right\}$$

is the p-variation of X. Whenever $p = 1$, we have the usual total variation.

Here is the main result of this paper.

THEOREM 1. *Let X_t, $t \geqslant 0$ be a real-valued Lévy process of almost surely unbounded total variation. Then the p-variation index $\upsilon(Y)$ of the corresponding OU-type process Y_t, $t \geqslant 0$ is equal to $\upsilon(X)$.*

3. Proofs

We begin by proving tw o auxiliary lemmas for real valued functions. Later we will apply them to the sample paths of an OU-type process.

LEMMA 2. *Let $f\colon [a,b] \to \mathbb{R}$ $(a < b)$ be a function of bounded total variation. Then for any function $g\colon [a,b] \to \mathbb{R}$ with a finite p-variation index $\upsilon(g)$ we have $\upsilon(f+g) \vee 1 = \upsilon(g) \vee 1$, where $x \vee y = \max\{x, y\}$.*

Proof. First consider the case $\upsilon(g) \leqslant 1$. Clearly, $\upsilon(g) \vee 1 = 1 \leqslant \upsilon(f+g) \vee 1$. Moreover, for any $p > 1$ we have $v_p(g) < +\infty$ and

$$v_p^{1/p}(f+g) \leqslant v_p^{1/p}(f) + v_p^{1/p}(g) < +\infty, \tag{4}$$

since $v_p^{1/p}(f) \leqslant v_1(f) < +\infty$ for any $p \geqslant 1$. And so $\upsilon(f+g) \leqslant p$. Letting $p \downarrow 1$ we obtain $\upsilon(f+g) \leqslant 1$ and $\upsilon(f+g) \vee 1 = 1$.

In the case $\upsilon(g) > 1$ we have $\upsilon(g) \vee 1 = \upsilon(g)$ and for any $p > \upsilon(g)$ by (4) we obtain $\upsilon(f+g) \leqslant p$. Now taking $p = p_n = \upsilon(g) + 1/n$ and letting $n \to \infty$ we get $\upsilon(f+g) \leqslant \upsilon(g)$. If $\upsilon(f+g) = \upsilon(g)$ we are done; otherwise, there exists a $p \in [1 \vee \upsilon(f+g), \upsilon(g))$ such that $v_p(f+g) < +\infty$. Then $g = (f+g) - f$ and

$$v_p^{1/p}(g) \leqslant v_p^{1/p}(f+g) + v_p^{1/p}(f) < +\infty,$$

a contradiction, since $v_p(g) = +\infty$. Therefore, $1 \vee \upsilon(f+g) \geqslant \upsilon(g)$, and the proof is completed.

Remark 3. It is obvious that the maximum with 1 cannot be omitted. As an example, take $f(x) = x$, $g(x) = 1 - x$, $x \in [0,1]$. Then $\upsilon(f) = \upsilon(g) = 1$ but $\upsilon(f+g) = 0$, since $f + g \equiv 1$.

Let $\|f\|_p := v_p^{1/p}(f) + \|f\|_\infty$, $p \in [1, +\infty)$.

LEMMA 4. *Let $f\colon [a,b] \to \mathbb{R}$ $(a < b)$ be a function of bounded total variation. Then for any function $g\colon [a,b] \to \mathbb{R}$ with the p-variation index $\upsilon(g) \geqslant 1$ we have*

$v(fg) \leqslant v(g)$. *If, in addition, g is of unbounded total variation, i.e., $v_1(g) = +\infty$, and the function $1/f$ is well-defined and of bounded total variation, then $v_1(fg) = +\infty$ and $v(fg) = v(g)$.*

Proof. Pick any $\varepsilon > 0$ and consider any $p \in (v(g), v(g) + \varepsilon)$. Then, by definition of $v(g)$, we get $v_p(g) < +\infty$. Since $p \geqslant 1$, applying a result of Krabbe [5] we obtain

$$\|fg\|_{[p]} \leqslant \|f\|_{[p]} \|g\|_{[p]} < +\infty,$$

and so $v_p(fg) < +\infty$. Moreover, $v(fg) \leqslant p < v(g) + \varepsilon$. Letting $\varepsilon \downarrow 0$, we obtain $v(fg) \leqslant v(g)$.

Now suppose that $v_1(g) = +\infty$ and $v_1(1/f) < +\infty$. If the function fg were of bounded total variation or of bounded p-variation for some $p < v(g)$ then, applying the above mentioned result of Krabbe, we would get

$$\|g\|_{[q]} \leqslant \|1/f\|_{[q]} \|fg\|_{[q]} < +\infty,$$

where we take $q = 1$ in the former case and $q = p$ in the latter case. But this is a contradiction in either case, since $v_1(g) = +\infty$ (applied in case $v(g) = 1$), and $v_q(g) = +\infty$ for any $q < v(g)$ (applied in case $1 < v(g)$). Therefore, $v(fg) \geqslant 1 = v(g)$ in case $v(g) = 1$, and $v(fg) \geqslant p$ for any $p < v(g)$ in case $v(g) > 1$. So either way we get $v(fg) \geqslant v(g)$, which completes the proof.

Remark 5. Without the assumption $v_1(g) = +\infty$ the equality $v(fg) = v(g)$ would no longer hold, in general, as one can take, for example, $f(x) = \mathrm{e}^{-x}$, $g(x) = \mathrm{e}^{x}$ for any $x \in [1, 2]$, so that $fg \equiv 1$ and $v(fg) = 0 < v(f) = v(g) = 1$.

We now proceed to the proof of the main theorem.

Proof of Theorem 1. Consider a Lévy process X_t of almost surely unbounded total variation and a corresponding OU-type process Y_t which we can write as

$$Y_t = \mathrm{e}^{-\lambda t} X_0 + \mathrm{e}^{-\lambda t} \int_0^t \mathrm{e}^{\lambda u} \, \mathrm{d}X_u = Z_t^{(1)} + \mathrm{e}^{-\lambda t} Z_t^{(2)}.$$

Clearly, the function $\mathrm{e}^{-\lambda t}$ and the process $Z_t^{(1)}$ are of bounded total variation on any finite interval $[a, b] \subset [0, \infty)$. So in order to show that $v(Y) = v(X)$ almost surely, by Lemmas 2 and 4, it suffices to show that $1 \leqslant v(X) = v(Z^{(2)})$ almost surely.

Since the function $h(u) = \mathrm{e}^{\lambda u}$, $u \in [0, T]$, is a continuous function of bounded total variation and the process X_u, $u \in [0, T]$, being a Lévy process is a semimartingale of bounded p-variation for any $p > 2$, by [4, Thm. 4.26] or [3, Thm. II.3.27] we get that the integral defining $Z_t^{(2)}$ exists in the Riemann–Stieltjes sense and by [4, Cor. 4.28] or [3, Prop. II.3.32] defines a function of bounded p-variation for any $p > v(X) \geqslant 1$. Hence, the inequality $v(Z^{(2)}) \leqslant v(X)$ holds almost surely.

To show that $v(Z^{(2)}) \geqslant v(X)$ consider any $p \geqslant 1$ such that $v_p(X) = +\infty$ almost surely. At least one such p exists, since X is assumed to be of unbounded total variation. For any $\omega \in \Omega$ such that $v_p(X(\omega)) = +\infty$ consider a sequence of partitions of $[0,t]$, say $\{\pi_n\}_{n=1}^{\infty}$, such that $v_p(X(\omega),\pi_n)\uparrow\infty$, as $n\to\infty$. Denote $\pi_n=\{t_i^n\colon\ 0=t_0^n<t_1^n<\cdots<t_{m_n}^n=t\}$ and $\Delta_{i,n}f=f(t_i^n)-f(t_{i-1}^n)$ so that

$$v_p(X(\omega),\pi_n)=\sum_{i=1}^{m_n}\left|\Delta_{i,n}X(\omega)\right|^p$$

and

$$v_p(Z^{(2)}(\omega),\pi_n)=\sum_{i=1}^{m_n}\left|\Delta_{i,n}Z^{(2)}(\omega)\right|^p=\sum_{i=1}^{m_n}\left|\int_{t_{i-1}^n}^{t_i^n}\mathrm{e}^{\lambda u}\,\mathrm{d}X_u(\omega)\right|^p.$$

Integration by parts (see [3, Thm. I.4.8]) yields:

$$\begin{aligned}\Delta_{i,n}Z^{(2)}(\omega)&=\mathrm{e}^{\lambda u}X_u(\omega)\big|_{u=t_{i-1}^n}^{u=t_i^n}-\int_{t_{i-1}^n}^{t_i^n}X_u(\omega)\lambda\mathrm{e}^{\lambda u}\,\mathrm{d}u\\&=\mathrm{e}^{\lambda t_i^n}\Delta_{i,n}X(\omega)+X_{t_{i-1}^n}(\omega)\Delta_{i,n}h-\int_{t_{i-1}^n}^{t_i^n}X_u(\omega)\lambda\mathrm{e}^{\lambda u}\,\mathrm{d}u,\end{aligned}$$

where $h(x)=\mathrm{e}^{\lambda x}$. By the triangle inequality,

$$\begin{aligned}\left|\Delta_{i,n}Z^{(2)}(\omega)\right|&\geqslant\mathrm{e}^{\lambda t_i^n}\left|\Delta_{i,n}X(\omega)\right|-|\Delta_{i,n}h|\left|X_{t_{i-1}^n}(\omega)\right|\\&\quad-\lambda(t_i^n-t_{i-1}^n)\sup_{t_{i-1}^n\leqslant u\leqslant t_i^n}\mathrm{e}^{\lambda u}\left|X_u(\omega)\right|\\&\geqslant\left|\Delta_{i,n}X(\omega)\right|-2\lambda\mathrm{e}^{\lambda t_i^n}(t_i^n-t_{i-1}^n)\sup_{t_{i-1}^n\leqslant u\leqslant t_i^n}\left|X_u(\omega)\right|,\end{aligned}$$

since $|\Delta_{i,n}h|\leqslant\lambda(t_i^n-t_{i-1}^n)\sup_{t_{i-1}^n\leqslant u\leqslant t_i^n}\mathrm{e}^{\lambda u}$. Letting $M(\omega)=\sup_{0\leqslant u\leqslant t}|X_u(\omega)|$, which is finite almost surely, since $X_u, u\in[0,t]$ is regulated, we get, for $p\geqslant 1$ as above,

$$\begin{aligned}\sum_{i=1}^{m_n}\left|\Delta_{i,n}X(\omega)\right|^p&\leqslant\sum_{i=1}^{m_n}\Big(\left|\Delta_{i,n}Z^{(2)}(\omega)\right|+2\lambda\mathrm{e}^{\lambda t}(t_i^n-t_{i-1}^n)M(\omega)\Big)^p\\&\leqslant 2^{p-1}\sum_{i=1}^{m_n}\Big(\left|\Delta_{i,n}Z^{(2)}(\omega)\right|^p+\big(2\lambda\mathrm{e}^{\lambda t}(t_i^n-t_{i-1}^n)M(\omega)\big)^p\Big).\end{aligned}$$

Since $\sum_{i=1}^{m_n}(t_i^n-t_{i-1}^n)^p\leqslant\max_i(t_i^n-t_{i-1}^n)^{p-1}\sum_{i=1}^{m_n}(t_i^n-t_{i-1}^n)\leqslant t^p$, and $v_p(X(\omega),\pi_n)\uparrow+\infty$, we obtain $v_p(Z^{(2)}(\omega),\pi_n)\uparrow+\infty$, as $n\to\infty$. Thus, $v_p(Z^{(2)})=+\infty$, so long as $v_p(X)=+\infty$. This implies $v(Z^{(2)})\geqslant v(X)$ almost surely.

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REZIUMĖ

M. Manstavičius. Ornšteino–Ulenbeko tipo procesų p-variacija

Straipsnyje nagrinėjama Ornšteino–Ulenbeko tipo procesų trajektorijų p-variacija. Įrodyta, kad bet kurio tokio proceso Y p-variacijos indeksas $v(Y)$ sutampa su šį procesą generuojančio Levy proceso X p-variacijos indeksu $v(X)$, jei X trajektorijos yra beveik tikrai neaprėžtos pilnosios variacijos.