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Normal form of formulas of pure hybrid logic

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Abstract. In this paper, we study a transformation of pure hybrid logic formulae, which do not have binding operator, into an equivalent normal form, which does not have any satisfiability operators in the scope of another satisfiability operator.

Keywords: pure hybrid logic, normal form.

1. Introduction

Lets extend multimodal logic with new symbols which we will call nominals. Elements of a new set NOM = $\{i, j, ...\}$ have to distinct from propositional symbols from set PROP = $\{p, q, ...\}$. Nominals are used to label nodes in the graph (elements of the structure set). Two nodes cannot be named with the same name. Nominals are counted as atomic formulae of the hybrid logics. Symbols $\top$, $\bot$ (*truth*, *false*) are also atomic formulae. Once we have names for the nodes, we can add another operator, which would allow us to declare that *formula F is valid at node i*. It is written as $@_i F$. By adding another operator $\downarrow$ we get hybrid logic.

Formula $\downarrow x.F$ has meaning: "F is valid in all nodes which are currently examined. Variable x scans through all nodes currently being examined."

We will look at formulae of pure hybrid logics, i.e., hybrid logics without propositional symbols. Such formulae contain just $\vee$, & and $\neg$. Sequents which we will look have form $\Gamma \vdash$, i.e., formulae will only be on the left hard side of $\vdash$. Symbol $\vdash$ will be implicit and skipped. There are various known methods for deriving valid formulae. C. Areces, H. de Nivelle, M. de Rijke [1] describes resolution method, P. Blackburn, M. Marx [3] – tableau method, T. Braüner [4] – natural deduction and sequent calculus, P. Blackburn [2] and J. Seligman [5] – sequent calculus. We will use sequent calculus H described in work [6]. We add new axiom to sequent calculus $\Gamma, @_s\bot$.

2. Transformation to the normal form

We will consider the formulae of hybrid logics without $\downarrow$ occurences.

Set of formulae $\mathcal{F}_n$ $(n = 0, 1, 2, ...)$ are defined as:

- $\mathcal{F}_0$ is the set of atomic formulae.
- $\mathcal{F}_{n+1} = \mathcal{F}_n \cup \{\alpha F\colon\ F \in \mathcal{F}_n \text{and } \alpha \text{ is } \neg,\ \Box_i,\ \Diamond_i, @_i\} \cup \{(F\ \beta\ G)\colon\ F,\ G \in \mathcal{F}_n \text{ and } \beta \text{ is } \&,\ \vee,\ \to\}$

DEFINITION 1. The depth of formula F we will call the smallest n, such that $F \in \mathcal{F}_n$.

LEMMA 1. *Following equivalences hold for hybrid logic formulae*:

- $\Box_j @_i F \equiv \Box_j \bot \vee @_i F$,
- $\Box_j (F \vee @_i G) \equiv \Box_j F \vee @_i G$,
- $\Box_j (F \& @_i G) \equiv \Box_j F \& \Box_j @_i G$.

Proof. We will only prove the first equivalence. Others can be proved similarly.

1. $@_s(\Box_j @_i F) \vdash @_s(\Box_j \bot \vee @_i F)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\oplus}{@_t\top,\ @_i\neg F,\ @_i F,\ @_s\Box_j @_i F,\ @_s\Diamond_j t}}{@_t\top,\ @_s@_i\neg F,\ @_t@_i F,\ @_s\Box_j @_i F,\ @_s\Diamond_j t}(SIMP)}{@_s\Box_j @_i F,\ @_s\Diamond_j t,\ @_t\top,\ @_s@_i\neg F}(\Box_j)}{@_s\Box_j @_i F,\ @_s\Diamond_j \top,\ @_s@_i\neg F}(\Diamond_j)}{@_s(\Box_j @_i F),\ \ @_s(\Diamond_j \top \& @_i\neg F)}(\&)$$

2. $@_s(\Box_j \bot \vee @_i F) \vdash @_s(\Box_j @_i F)$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\genfrac{}{}{0pt}{}{\oplus}{@_t l,\ @_t\neg l,\ @_s\Box_j\bot,\ @_s\Diamond_j t,\ @_i\neg F}}{@_t(l\&\neg l),\ @_s\Box_j\bot,\ @_s\Diamond_j t,\ @_i\neg F}(\&)}{@_t\bot,\ @_s\Box_j\bot,\ @_s\Diamond_j t,\ @_i\neg F}}{@_s\Box_j\bot,\ @_s\Diamond_j t, @_i\neg F}(\Box_j) \qquad \dfrac{\genfrac{}{}{0pt}{}{\oplus}{@_i F,\ @_s\Diamond_j t,\ @_i\neg F}}{@_s@_i F,\ @_s\Diamond_j t,\ @_i\neg F}(SIMP)}{@_s(\Box_j\bot \vee @_i F),\ @_s\Diamond_j t,\ @_i\neg F}(\vee)}{@_s(\Box_j\bot \vee @_i F),\ @_s\Diamond_j t,\ @_t@_i\neg F}(SIMP)}{@_s(\Box_j\bot \vee @_i F),\ @_s(\Diamond_j @_i\neg F)}(\Diamond_j)$$

End of proof.

LEMMA 2. *Following equivalences hold for hybrid logic formulae*:

- $\Diamond_j @_i F \equiv \Diamond_j \top \& @_i F$,
- $\Diamond_j (F \& @_i G) \equiv \Diamond_j F \& @_i G$,
- $\Diamond_j (F \vee @_i G) \equiv \Diamond_j F \vee \Diamond_j @_i G$,
- $@_s @_i F \equiv @_i F$,
- $@_s (F \vee @_i G) \equiv @_s F \vee @_i G$,
- $@_s (F \& @_i G) \equiv @_s F \& @_i G$.

Proof. We will only prove the first equivalence. Others can be proved similarly. From the first lemma we have that $\Box_j @_i G \equiv \Box_j \bot \vee @_i G$. Consequently

$$\neg(\Box_j @_i G) \equiv \neg(\Box_j \bot \vee @_i G)$$

$$\Diamond_j @_i \neg G \equiv \Diamond_j \top \& @_i \neg G$$

$$\Diamond_j @_i F \equiv \Diamond_j \top \& @_i F$$

End of proof.

THEOREM 1. *Any formula from pure hybrid logics without ↓ entries can be transformed to normal form*:

$$F \equiv \bigwedge_{j=1}^{r} \Big(\bigvee_{i \in NOM(F)} @_i G_{j,i} \vee H_j \Big), \quad @ \notin G, H.$$

Proof. We will prove this using mathematical induction on the depth of the formula.

The base of induction. Depth of formula is equal to zero. Such formula is atomic so it is already in the normal form.

Induction hypothesis. Say that theorem is correct for all formulae with depth not greater than n.

Inductive step. We will prove that theorem is correct for all formulae with depth $n+1$. The examined formula F can have following forms: 1. $F = \neg A$, 2. $F = \Box_l A$, 3. $F = \Diamond_l A$, 4. $F = A \& B$, 5. $F = A \vee B$, 6. $F = A \to B$, 7. $F = @_l A$.

Since A and B has depth not greater than n then theorem holds for them according induction hypothesis.

First case. $F = \neg A$. According induction hypothesis A can be transformed into normal form:

$$A \equiv \bigwedge_{j=1}^{r} (\bigvee_{i \in NOM(F)} @_i G_{j,i} \vee H_j \Big), \quad @ \notin G, H,$$

$$F = \neg \Big(\bigwedge_{j=1}^{r} \Big(\bigvee_{i \in NOM(F)} @_i G_{j,i} \vee H_j \Big)\Big) = \bigvee_{j=1}^{r} \Big(\bigwedge_{i \in NOM(F)} @_i \neg G_{j,i} \& \neg H_j \Big).$$

By using rules of distributivity (from CNF we can convert to DNF and vise versa) we get:

$$F = \bigwedge_{j'=1}^{r'} \Big(\bigvee_{i \in NOM(F)} @_i \neg G_{j',i} \vee \neg H_{j'} \Big)$$

$$= \bigwedge_{j'=1}^{r'} \Big(\bigvee_{i \in NOM(F)} @_i G'_{j',i} \vee H'_{j'} \Big), \quad @ \notin G', H'.$$

Note. $@_i A \vee @_i B \equiv @_i (A \vee B)$

Second case. $F = \Box_l A$. According induction hypothesis A has normal form:

$$A \equiv \bigwedge_{j=1}^{r} \Big(\bigvee_{i \in NOM(F)} @_i G_{j,i} \vee H_j \Big), \quad @ \notin G, H,$$

$$F = \Box_l\Big(\bigwedge_{j=1}^{r}\Big(\bigvee_{i\in NOM(F)} @_i G_{j,i} \vee H_j\Big)\Big)$$

By using equivalence $\Box_l(A\&B) \equiv \Box_l A\&\Box_l B$ and Lemma 1:

$$F = \bigwedge_{j=1}^{r}\Big(\bigvee_{i\in NOM(F)} @_i G_{j,i} \vee \Box_l H_j\Big)$$

$$= \bigwedge_{j=1}^{r}\Big(\bigvee_{i\in NOM(F)} @_i G_{j,i} \vee \Box_l H'_j\Big), \quad H'_j = \Box_l H_j, \ \ @ \notin G, H$$

Third case. $F = \Diamond_l A$. According induction hypothesis A has normal form:

$$A \equiv \bigwedge_{j=1}^{r}\Big(\bigvee_{i\in NOM(F)} @_i G_{j,i} \vee H_j\Big), \quad @ \notin G, H.$$

Using equivalence $\Diamond_l(A\&B) \equiv \Diamond_l A\&\Diamond_l B$ and Lemma 2:

$$F = \bigwedge_{j=1}^{r}\Big(\Diamond_l\Big(\bigvee_{i\in NOM(F)} @_i G_{j,i} \vee H_j\Big)\Big) = \bigwedge_{j=1}^{r}\Big(\bigvee_{i\in NOM(F)} \Diamond_l @_i G_{j,i} \vee \Diamond_l H_j\Big)$$

Every formula $\Diamond_l @_i G_{j,i}$ can be replaced with $\Diamond_l \top\&@_i G_{j,i}$ (Lemma 2):

$$F = \bigwedge_{j=1}^{r}\Big(\bigvee_{i\in NOM(F)} (\Diamond_l \top\&@_i G_{j,i}) \vee \Diamond_l H_j\Big) =$$

$$= \bigwedge_{j=1}^{r}\Big(\Diamond_l \top\& \bigvee_{i\in NOM(F)} @_i G_{j,i} \vee \Diamond_l H_j\Big) =$$

$$= \bigwedge_{j=1}^{r}\Big(\Big(\Diamond_l \top \vee \Diamond_l H_l\Big)\&\Big(\bigvee_{i\in NOM(F)} @_i G_{j,i} \vee \Diamond_l H_j\Big)\Big) =$$

$$= \bigwedge_{j=1}^{r'}\Big(\bigvee_{i\in NOM(F)} @_i G'_{j,i} \vee H'_j\Big)$$

Other cases can be proved similarly. End of proof.

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REZIUMĖ

D. Aleknavičiūtė, S. Norgėla. Hibridinės logikos formulių normalioji forma

Aprašomas grynosios hibridinės logikos formulių be suvaržymo operatoriaus transformavimo metodas į ekvivalenčią normaliąją formą, kurioje įvykdomumo operatorių veikimų srityse nėra įvykdomumo operatorių įeičių.