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Integral extensions in Procesi category

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In this paper an *integral extensions* in Procesi category are defined and theorems for ideals lying over left strongly prime ideals are proved.

All rings considered are associative with identity element which should be preserved by ring homomorphisms, all modules are unitary. $A \subset B$ means that A is the proper subset of B.

Let φ: $R \to S$ be a ring homomorphism. Then S becomes a canonical R-bimodule and we write rs and sr instead of $\varphi(r)s$ and $s\varphi(r)$ for $r \in R$, $s \in S$. Let $Z_S(R) = \{x \in S \mid rx = xr,\ \forall r \in R\}$ be the set of R-centralizing elements of the ring S.

We call φ a *centred homomorphism* and S a *centred extension* of R via φ, provided $S = RZ_S(R)$ (see [3]). This means that $s = \sum_k r_k x_k$ for each element $s \in S$ with some $r_k \in R$ and $x_k \in Z_S(R)$. If $Z_S(R)$ is commutative then centred extension is called a *central extension*. Rings and their centred homomorphisms form a category, known as *Procesi category* (see [5]). We recall that a centred extension $R \subseteq S$ is called a *liberal extension* if S is finitely generated as a canonical R-module.

Let φ: $R \to S$ be a centred homomorphism of rings. We say that φ is an *integral homomorphism* if every finite subset $\{s_1, \dots, s_n\} \subseteq S$ is contained in some subring $A \subseteq S$ which is liberal extension of the ring $\varphi(R)$, i.e., A is finitely generated as a canonical R-module by a finite subset of R-centralizing elements. In this case S is called an integral extension of R via φ. When R and S are commutative the given definition is equivalent to the classical by Proposition 1, Ch. 5 in [2]. It is clear that integral extensions of the field precisely are locally finite algebras. S is integral over R if and only if S is an inductive limit of the liberal extensions over $\varphi(R)$ (see [3]).

Let M be an R-bimodule, denote by $Z_M = Z_M(R) = \{x \in M \mid rx = xr,\ \forall r \in R\}$. Consider a system of n homogeneous equations $\sum_l r_{kl} x_l = 0$, where $r_{kl} \in R$, $x_l \in Z_M$, $1 \leqslant k, l \leqslant n$.

If $X = (r_{kl}) \in \mathbf{M}_n(R)$ is a matrix of coefficients, denote by

$$d = det X = \sum_{\sigma} (-1)^{\sigma} r_{1\sigma(1)} \cdots r_{n\sigma(n)}$$

the formal analogue of the determinant for noncommutative ring R. In 5.1 of [1] M. Artin pointed out that, like in the commutative case, $dx_k = 0$, for $1 \leqslant k \leqslant n$.

A left non-zero module M over the ring R is called *strongly prime* if for any non-zero elements $x, y \in M$, there exists a finite set of elements $\{\alpha_1, \dots, \alpha_n\} \subseteq R$, $n = n(x, y)$, such that $Ann_R\{\alpha_1 x, \dots, \alpha_n x\} \subseteq Ann_R\{y\}$.

A submodule N of some left R-module M is called strongly prime if the quotient module M/N is strongly prime R-module. Particularly, we obtain the notion o the left strongly prime ideal $\mathfrak{p} \subset R$. See [3, 4, 5] for the properties of the left strongly prime ideals and modules.

THEOREM 0.1. *Let $R \subseteq S$ be a centred integral extension. Then for each left maximal ideal $\mathfrak{m} \subset R$ there exists left maximal ideal $\mathfrak{q} \subset S$ lying over $\mathfrak{m}$, i.e., such that $\mathfrak{q} \cap R = \mathfrak{m}$.*

Proof. We prove first that the extension of $\mathfrak{m}$ in S is proper left ideal in S, i.e., that $\mathfrak{m}^e = S\mathfrak{m} \neq S$.

Evidently, $\mathfrak{m}^e = S\mathfrak{m} = Z_S(R)\mathfrak{m}$. If $\mathfrak{m}^e$=S, we could find a liberal extension $A \subseteq S$ of the ring R and the expression

$$\tilde{r}_1 x_1 + \cdots + \tilde{r}_n x_n = 1$$

with $\tilde{r}_k \in \mathfrak{m}$, $x_k \in Z_A(R)$, $1 \leqslant k \leqslant n$, and elements x_k generate A as R-module. Multiplying this equality by R-centralizing elements x_k, we obtain the system of homogeneous equations

$$\sum_l (r_{kl} - \delta_{kl}) x_l = 0 \quad \text{with} \quad 1 \leqslant k,\ \ l \leqslant n,\ \ r_{kl} \in \mathfrak{m}.$$

Then, by Artin's remark, $dx_k = 0$ for all $1 \leqslant k \leqslant n$, so $d = 0$ because $1 \in A$. But $d = (-1)^n + \alpha$, where $\alpha \in \mathfrak{m}$. A contradiction, so $\mathfrak{m}^e \neq S$.

By Zorn's lemma, there exists some left maximal ideal $\mathfrak{q} \subseteq S$ containing $\mathfrak{m}^e$. Because $\mathfrak{m} \subseteq \mathfrak{q} \cap R \neq R$ and $\mathfrak{m}$ is maximal, we have that $\mathfrak{q} \cap R = \mathfrak{m}$.

THEOREM 0.2. *Let $R \subseteq S$ be a centred integral extension. Then for each left strongly prime ideal $\mathfrak{p} \subset R$ there exists some strongly prime ideal $\mathfrak{q} \subset S$ lying over $\mathfrak{p}$.*

Proof. By Corollary 1.5 in [4], there exist a ring of polynomials $R\langle X_I \rangle$ with the set of noncommuting indeterminates $X_\alpha \in X_I$, $\alpha \in I$, and some left maximal ideal $\mathfrak{M} \subset R\langle X_I \rangle$, such that $\mathfrak{p} = \mathfrak{M} \cap R$.

It is easy to see that the ring $S\langle X_I \rangle$ is the centred integral extension of the ring $R\langle X_I \rangle$. By Theorem 1, there exists left maximal ideal $\mathfrak{M}_1 \subset S\langle X_I \rangle$ lying over $\mathfrak{M}$, i.e., such that $\mathfrak{M} = \mathfrak{M}_1 \cap R\langle X_I \rangle$. It is well known that preimage of the left strongly prime ideal under centred homomorphism is strongly prime. So $\mathfrak{q} = \mathfrak{M}_1 \cap S$ is left strongly prime ideal lying over $\mathfrak{p} \subset R$.

References

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REZIUMĖ

A. Kaučikas. Sveikieji plėtiniai Pročezio kategorijoje

Įrodyta, kad sveikuosiuose Pročezio kategorijos plėiniuose kiekvieną stipriai pirminį vienpusį idealą galima pakelti.