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Sequent calculus Sk4 for skolemized formulas

Andrejus BELOVAS, Stanislovas NORGĖLA (VU)
e-mail: andrejus@dtiltas.lt, stasys.norgela@maf.vu.lt

We will consider the formulas of quantified modal logic, which contain logical connectives $\neg$, &, $\vee$ and the negation occurs only in front of atomic formulas. This means that the considered formulas are in normal negation form. We will consider two ways of skolemization: Outer and Inner (refer to A.Nonnengart [1] in case of classical logic).

DEFINITION 1. We obtain the skolemization of a formula F by replacing every occurrence of sub-formulas $\exists x \Phi$ in F with a corresponding $\Phi(f(\bar{y_n}/x)$, where f is an n-place function symbol which is new to the problem under consideration. We speak of Outer skolemization in case the variables $\bar{y_n}$ are all the universally quantified variables such that $\exists x \Phi$ is a subformula of a $\forall y_i G$ for each $1 \leqslant i \leqslant n$. If the variables $\bar{y_n}$ are all the free variables in the sub-formula $\exists x \Phi$, we speak of Inner skolemization.

We will use Inner skolemization for the formulas of modal logic S4, as it was described in M. Cialdea's work [2]. We will not assign degrees to the terms of the initial (while being not skolemized) formula. We will denote skolemized F as $Sk(F)$. Before skolemizing any formula F it is possible:

- to move all quantifiers outwards (using equivalence formulas of classical logic) so that all quantifiers appear as close to the beginning of a formula as possible (we will denote such formula as F_O),
- to move all quantifiers inwards as deep as it is possible (we will denote such formula as F_I).

EXAMPLE. $F = \forall x \Box \exists y (P(x,y) \vee \exists z \Diamond \forall u \exists v (P(z,u) \vee Q(v)))$.
Then
$F_O = \forall x \Box \exists y \exists z (P(x,y) \vee \Diamond \forall u \exists v (P(z,u) \vee Q(v)))$,
$F_I = \forall x \Box (\exists y P(x,y) \vee \exists z \Diamond (\forall u P(z,u) \vee \exists v Q(v)))$.

We assume that there are no two bound variables in a formula with the same name.

It is known that F_O and F_I are equal in the calculus S4. Let us now define the calculus Sk4 and prove, that $F_I \vdash$ is derivable in the calculus S4 if and only if the sequent $Sk(F_I) \vdash$ is derivable in the calculus Sk4. Similarly $F_O \vdash_{S4}$ if and only if $Sk(F_O) \vdash_{Sk4}$. From this it will follow that $Sk(F_O)$ and $Sk(F_I)$ are deductively equivalent. Apealing to that fact we will prove the decidability of one class. The calculus Sk4 has it's own independent meaning as well.

The calculus Sk4. Axioms: $F, \neg F, \Gamma \vdash$
Rules:

$$(\&)\quad \frac{F, G, \Gamma \vdash}{F \& G, \Gamma \vdash} \qquad (\vee)\quad \frac{F, \Gamma \vdash \quad G, \Gamma \vdash}{F \vee G, \Gamma \vdash}$$

In case, when during skolemization all quantifiers were moved outwards, the rules are applied if there are no zero-degree terms, included in both formulas F, G.

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Substitution rules:

$$(S1)\quad \frac{F(t/x^0), F(x^0), \Gamma \vdash}{F(x^0), \Gamma \vdash}, \qquad (S2)\quad \frac{F(a/f_x^0(t_1, ..., t_n)), \Gamma \vdash}{F(f_x^0(t_1, ..., t_n)), \Gamma \vdash}, \qquad (S3)\quad \frac{F(b), \Gamma \vdash}{F(b^0), \Gamma \vdash},$$

where $t, t_1, ..., t_n$ are ground terms, in which all subterms don't have any degrees; a is a new individual constant.

Modal rules:

$$(\Box)\quad \frac{F', \Box F, \Gamma \vdash}{\Box F, \Gamma \vdash}, \qquad (\Diamond)\quad \frac{F', \Gamma^* \vdash}{\Diamond F, \Gamma \vdash},$$

where formula F' is obtained from the formula F, reducing all its terms' degrees, lager than zero, by one. Γ^* − all Γ formulas, beginning with the modal operator $\Box$. Moreover, there are no zero degree terms in the conclusions of the rules (that is, in each sequent under the line).

THEOREM 1. *Sequent $F_I \vdash$ ($F_O \vdash$) is derivable in the calculus S4 if and only if the sequent $Sk(F_I) \vdash$ ($Sk(F_O) \vdash$ respectively) is derivable in the calculus Sk4.*

Proof. We will construct the derivation tree of the sequent $Sk(F_I) \vdash$ considering the derivation tree of the sequent $F_I \vdash$ (the derivation tree of the sequent $Sk(F_O) \vdash$ will be constructed in the same way). Following the path from the bottom to the top, for every sequent's $F_I \vdash$ derivation rule we will find the corresponding rule of the calculus Sk4. The rule ($S1$) corresponds to the rule ($\forall \vdash$), the rules ($S2$) and ($S3$) correspond to the rule ($\exists \vdash$). Following the path from the bottom to the top we replace the applications of the rule ($\forall \vdash$)

$$\frac{G(t), \forall x G(x), \Gamma \vdash}{\forall x G(x), \Gamma \vdash}$$

by the applications of the rule ($S1$)

$$\frac{G'(t), G'(x^0)), \Gamma' \vdash}{G'(x^0), \Gamma' \vdash}.$$

We replace the applications of the rule ($\exists \vdash$)

$$\frac{G(a), \Gamma \vdash}{\exists x G(x), \Gamma \vdash}$$

by the applications of the rule ($S2$) (or the rule ($S3$))

$$\frac{G'(a), \Gamma' \vdash}{G'(f_x^0(\bar{y_n})), \Gamma' \vdash},$$

where $G' = Sk(G)$ and $\Gamma' = Sk(\Gamma)$. Note that in the derivation in calculus Sk4 axioms will remain the same, as in the derivation in calculus S4. Besides, the rules for logical operations are the same. The applications of ($\Box \vdash$) and ($\Diamond \vdash$) rules are replaced by the corresponding rules in Sk4.

In fact, the degrees in skolemized formulas show the number of modal operators in the initial formula's prefix, appearing before the quantifier, which bounds the corresponding variable. If the degree appeared after the skolemization of a formula, it means that the formula begins with a quantifier. And if this degree $n \geqslant 1$, it shows that the formula after some skolemization steps was obtained from the initial formula, in which investigated quantifier was in the scope of exactly n modal operators. Since in the applications of the rules ($\Box \vdash$) and ($\Diamond \vdash$) one modal operator is been eliminated, the degrees of all terms in skolemized formulas, which are larger than 0, are reduced by one. It is noteworhty, that if we were applying modal rules step by step, and in the result we obtained the formula, which doesn't have a zero skolemization degree, it means that the initial skolemization was performed incorrectly. The premises and the conclusions in the applications of the substitution and modal operator's rules in calculus Sk4 are deductively equal to the corresponding premises and conclusions in calculus S4.

Assume that $Sk(F_I) \vdash$ is deducible in calculus Sk4. We can construct such a derivation tree, that if in formula F there is a subformula having the shape of $\forall z_1 ... \forall z_n G(z_1, ..., z_n)$ and $G(t_1, ..., t_n)$ is obtained by applying n times the rule ($S1$) to the formula $G(z_1^i, .., z_n^i)$, then in the derivation tree the applications of the rule are met in the same way (following the path from the bottom to the top), that is, at first ($S1$) is applied to z_1^0, then to $z_2^0, ..., z_n^0$. This can be done, because the terms, we replaced z_j^0 with, are ground terms. We can construct such a derivation tree, that the elimination order would stay the same to the subformulas having the shape of $\exists z_1 ... \exists z_n G(z_1, ..., z_n)$ as well.

Let us show, how the derivation of $F_I \vdash$ in calculus S4 can be obtained, having the derivation tree of $Sk(F_I) \vdash$. Consider the first application of a rule. It could be: 1) the application of a logical operation rule, 2) one of substitution rules ($S1$), ($S2$), ($S3$), 3) a modal rule. In the first case there are no zero-degree terms in formula $Sk(F)$, included in both formulas. It means, that the main logical operation of a formula F is either conjunction or disjunction. So we apply the same rule in the derivation tree of the sequent $F \vdash$.

In the second case we have the application of either the rule ($S1$) or ($S3$), because the rule ($S2$) can be applied only if before that the rule ($S1$) was applied m-times (m is the number of zero-degree variables in term $f^0(t_1, ..., t_n)$). We replace the application of the rule ($S1$) (or ($S3$)) by the application of the rule ($\forall \vdash$) with the same ground term (in case of the rule ($S3$) it would be replaced by the application of the rule ($\exists \vdash$)).

In the third case the same rules are being applied, with the only difference – the reducing of degrees operation is not performed.

The replacements in other cases are made likewise. Theorem is proved.

DEFINITION 2. Modal literals are the expressions of the form l, $\Box l$, $\Diamond l$, where l is a literal of classical logic.

DEFINITION 3. The calculus S4 will be called minus-normal if the rule

$$\frac{A(a), \forall x A(x), \Gamma \vdash}{\forall x A(x), \Gamma \vdash}$$

contains the variable a in its conclusion as a free variable (if the conclusion does not contain free variables, then a is some free variable).

It is proved in [3] that if there are no functional symbols in a formula F, then it is deducible in the calculus S4 if and only if it is deducible in the minus-normal S4.

Consider the formulas without functional symbols (constants may be included), having the shape of

$$\Box \forall x_1 ... \forall x_n \exists y_1 ... \exists y_m (D_1 \vee ... \vee D_s), \tag{1}$$

where D_i $(i = 1, ..., s)$ is the disjunction of modal literals. Every existentially quantified individual variable y_i $(i = 1, ..., m)$, must be included only in one clause, that is it can be met only in one clause of the matrix. Moreover, every y_i occurs only in predicate variables, where except y_i only constants can be met. In particular, if there are only one-place predicate variables and the variables y_i occur only in one of the clauses, such formulas do belong to the class under consideration.

THEOREM 2. *The class of formulas, having the shape of* (1) *is decidable.*

Proof. Formula F is derivable in S4 if and only if $Sk(F_I) \vdash$ is derivable in calculus Sk4. There are no functional symbols in formula $Sk(F_I)$, as during the skolemization $y_1, ..., y_m$ are replaced by the first-degree constants (suppose, they are $a_1^1, ..., a_m^1$). Since the minus-normality is valid, the derivations of formulas without functional symbols, so in the derivation, applying the rule $(S1)$, we can replace the variables x_i^0 only by the constants from the initial formula, or $a_1^0, ..., a_m^0$. There is a finite number of such substitutions available. Thus after the finite number of steps we will either find the derivation of $Sk(F_I) \vdash$, or be persuaded that a derivation does not exist. Theorem is proved.

References

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2. M. Cialdea, Resolution for some first-order modal systems, *Theoretical Computer Science*, **85**, 213–229 (1991).
3. S. Norgela, Some decidable classes of formulas of modal logic S4, *Liet. matem. rink.*, **41** (spec. issue), 408–412 (2001).

REZIUMĖ

A. Belovas, S. Norgėla. Sekvencinis skaičiavimas Sk4 skulemizuotoms formulėms

Nagrinėjami du skulemizavimo modalumo logikos formulėms būdai: išorinis ir vidinis. Įrodoma, kad skulemizuotos skirtingais būdais formulės yra deduktyviai ekvivalenčios. Remiantis aprašytu skaičiavimu skulemizuotoms formulėms, įrodomas vienos klasės išsprendžiamumas.