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Modeling of zonal anisotropic variograms*

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Abstract. Most standard methods of geostatistical analysis are built upon the basic assumption of isotropy. However, spatial data in most cases are not isotropic. In practice typically we find data with zonal anisotropy. Zonal anisotropy is more complex than geometric; what is more, most software requires anisotropy to be geometric. In this paper an overview of models of zonal anisotropy is presented. Models of variograms with zonal anisotropy were fitted to the salinity data using *R*, the system for statistical computations and analysis.

Keywords: variogram, range, sill, nugget effect, zonal anisotropy.

1. Introduction

The variogram is critical input to geostatistical studies: it is a tool to investigate and quantify the spatial variability. Most geostatistical estimation or simulations algorithms require an analytical variogram model. Most standard methods of geostatistical analysis are built upon the basic assumption of isotropy. Isotropy means that the process operates identically in every direction. However, spatial data in most cases are not isotropic: the existence of natural barriers, of social and economic trends, of directional differences in communications, shows that spatial processes might vary with direction. A variogram or covariance function is anisotropic if it changes in some way with respect to direction, i.e., the spatial correlation depends on direction.

For spatial locations $\{s_i\colon\ s_i \in D \subset R^d\}$ in region D, suppose we observe responses $Z(s_i),\ i = 1, \dots, N$, where $Z = (Z(s_1), Z(s_2), \dots, Z(s_N))'$ is viewed as a single observation from a random field. Under the intrinsic hypothesis of Matheron (1963),

$$E\big(Z(s_1) - Z(s_2)\big) = 0, \quad var\big(Z(s_1) - Z(s_2)\big) = 2\gamma(s_1 - s_2) = 2\gamma(h), \qquad (1)$$

where $h = s_1 - s_2$ is separation vector, $2\gamma(h)$ is called the variogram and $\gamma(h)$ is the semivariogram. A stronger assumption is that the process $Z(s)$ is second-order or weakly stationary, in witch case

$$E\big(Z(s)\big) = \mu, \quad cov\big(Z(s_1), Z(s_2)\big) = C(s_1 - s_2) = C(h) < \infty. \qquad (2)$$

Classical semivariogram has the following shape as shown in Fig. 1.

Range is the distance at which the semivariogram becomes a constant. If $\lim_{|h|\to\infty} \gamma(|h|) = \gamma_\infty < \infty$, then γ_∞ is called sill of semivariogram. Nugget effect

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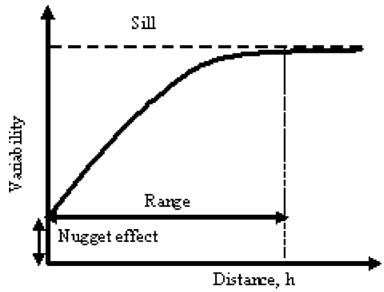

Fig. 1. Semivariogram representation.

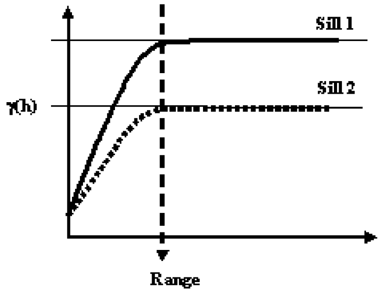

Fig. 2. Zonal anisotropy.

shows the pure random variation in population density or it may be associated with sampling error [2].

There are two types of anisotropy: geometric and zonal anisotropy. Overview of anisotropy types was given in [3]. Geometric anisotropy occurs when the range, but not the sill, of the variogram changes in different directions [1]. Zonal anisotropy exists when sill of variogram changes with direction (see Fig. 2).

Anisotropy in the data can be investigated by computing sample variograms in different directions. Differences in sample variograms could be an indication of anisotropy. When the experimental variograms show different behaviours in different directions an anisotropic variogram model should be fitted to them. There are however some practical difficulties in fitting the most appropriate model. These difficulties can lead to a wrong model of the true underlying structure. When the directional variograms show different variability for the different directions the sills are not comparable. In such a case geostatistics can be used to establish a variogram model made up of components having a so called zonal anisotropy.

Geometric anisotropy is typically dealt with by simply transforming the coordinates [3]. In turn zonal anisotropy is more complex than geometric and software almost always requires anisotropy to be geometric.

Zonal anisotropy is indicated by directional variograms that have the same range but different sills. Pure zonal anisotropy is usually not seen in practice; typically it is found in combination with geometric anisotropy. In order to model this mixture of anisotropic variograms, the overall variogram is set to a weighted sum of individual models of the directional variograms scaled by their ranges. In this process, called nesting, the choice of weights requires a trial and error approach with a constraint that the sum of the weights equals the sill of the overall variogram.

There are few ways for modelling semivariograms with zonal anisotropy:

1. Anisotropic semivariogram can be expressed as sum of individual models
$$\gamma(h_u, h_v) = \gamma_1(h_u) + \gamma_2(h_v), \tag{3}$$
where u and v represents the two main directions of anisotropy, $\gamma_1(h_u)$ and $\gamma_2(h_v)$ are semivariogram values in directions showing maximum and minimum continuity, so two directions with minimum and maximum ranges must be identified and fitted independently. This model fits well the two main directions, but this model is not satisfactory along intermediate directions, because in this model the total sill is the sum of the component sills in both directions.
2. The right approach to model all directions consistently with the observations made on the two main directions of variability will consist in fitting a model with two components: the first component is isotropic and the second zonal anisotropic.
$$\gamma(h_u, h_v) = \gamma_1\left(\sqrt{h_u^2 + h_v^2}\right) + \gamma_2(h_v). \tag{4}$$
The sill of the second component is equal to the difference between the sills observed along the main directions of continuity. In this model the total sill lies between the sill of the isotropic component and the sill fitted to the variogram in the direction of maximum variability.
3. In this model marginal variograms are used. $\gamma(h_u, h_v)$ has two marginals $\gamma(h_u, 0)$, $\gamma(0, h_v)$. Since $\gamma_u(0) = \gamma_v(0) = 0$
$$\gamma(h_u, 0) = a\gamma_u(h_u), \quad \gamma(0, h_v) = b\gamma_v(h_v).$$
Thus $\gamma(h_u, h_v)$ can be rewritten in terms of these marginals. So in the case of two components the result is considerably simplified
$$\gamma(h_u, h_v) = \gamma(h_u, 0) + \gamma(0, h_v) - K\gamma(h_u, 0) \times \gamma(0, h_v), \tag{5}$$
where $0 < K \leqslant 1/(\max(Sill_1, Sill_2))$.

2. Example

Salinity data, collected in the coastal zone of the Baltic sea, were used to model semivariograms with zonal anisotropy. All calculations have been done using package *Gstat*, which is a part of *R* package – the language and environment for statistical computing and graphics.

First of all we need to identify anisotropy, therefore we calculated variograms for many directions (with tolerance angle). In this case we used Function *Variogram*(y, *locations*, X, *cutoff*, *width*, *alpha*, *beta*, *tol.hor*, *tol.ver*) for defining direction in plane (x, y) (argument *alpha*), horizontal tolerance angle (arguments *tol.hor*). Using function *Fit.variogram*(*object*, *model*) we fitted parametric anisotropic spherical model and were looking for differences between sills and ranges. In our investigation we found out that sills and ranges differ in different directions, so we come to a conclusion that in this case we have zonal anisotropy.

To model semivariogram in the way (3) first thing is to identify the 2 directions showing the minimum and maximum variability. This could be done analyzing experimental directional semivariograms. We found out that semivariogram in direction with maximum range is

$$\gamma_1(h_u) = 0.040\left(\frac{3h_u}{2\cdot 62208} - \frac{1}{2}\left(\frac{h_u}{62208}\right)^3\right). \tag{6}$$

Semivariogram in direction with minimum range is

$$\gamma_2(h_v) = 0.030\left(\frac{3h_v}{2\cdot 36000} - \frac{1}{2}\left(\frac{h_v}{36000}\right)^3\right). \tag{7}$$

So, the resulting model has the following form:

$$\begin{aligned}\gamma(h_u,h_v) &= \gamma_1(h_u) + \gamma_2(h_v)\\ &= 0.040\left(\frac{3h_u}{2\cdot 62208} - \frac{1}{2}\left(\frac{h_u}{62208}\right)^3\right) + 0.030\left(\frac{3h_v}{2\cdot 36000} - \frac{1}{2}\left(\frac{h_v}{36000}\right)^3\right).\end{aligned} \tag{8}$$

Semiovariogram equation of model (4) has the following form:

$$\begin{aligned}\gamma(h_u,h_v) &= \gamma_1\left(\sqrt{h_u^2 + h_v^2}\right) + \gamma_2(h_v)\\ &= 0.035\left(\frac{3h}{2\cdot 51840} - \frac{1}{2}\left(\frac{h}{51840}\right)^3\right) + 0.010\left(\frac{3h}{2\cdot 36000} - \frac{1}{2}\left(\frac{h}{36000}\right)^3\right),\end{aligned}$$

where the first component is isotropic and the second – zonal anisotropic.

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REZIUMĖ

L. Budrikaitė. Negeometrinė erdvinių variogramų anizotropija

Šiame straipsnyje aprašytas negeometrinės (zoninės) anizotropijos tipas bei semivariogramų su zonine anizotropija modeliavimo būdai. Naudojantis nemokamos, statistiniams skaičiavimams skirtos sistemos *R* paketu *Gstat* , užrašyti du semivariogramų modeliai Baltijos jūros druskingumo duomenims.