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Explicit formulas in asymptotic expansions for Euler's approximations of semigroups

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1. Introduction and results

Let X be a complex Banach algebra with norm $\|\cdot\|$. A family $S(t)$, $t > 0$ of elements of a Banach algebra X is called a semigroup if $S(t+s) = S(t)S(s)$, for all $t, s > 0$ (see[8]). We define the resolvent $R(\lambda)$, $\lambda \in \mathbb{C}$ of the semigroup $S(t)$ as the Laplace transform $R(\lambda) = \int_0^\infty \mathrm{e}^{-\lambda t} S(t)\,\mathrm{d}t$. We also define the functions $t \mapsto E_n(t) = R^n(n/t)(n/t)^n$, $n \in \mathbb{N}$, called the Euler approximations of semigroup $S(t)$.

In [2] Bentkus obtained asymptotic expansions for Euler's approximations of semigroups with explicit and optimal bounds for the remainder terms. The approach was based on applications of the Fourier-Laplace transforms and a reduction of the problem to the convergence rates and asymptotic expansions in the Law of Large Numbers.

In this paper we provide explicit formulas in asymptotic expansions for Euler's approximations of semigroups.

First we introduce some additional notation. Henceforth $\sum_{i_1+\ldots+i_n=k}$ means summation over all distinct ordered n-tuples of positive integers $i_1, \ldots, i_n$ whose elements sum to k. Write

$$c_{k,j} = \frac{1}{j!} \sum_{i_1+\ldots+i_j=k+j} \frac{1}{i_1 i_2 \ldots i_j}, \tag{1.1}$$

where $i_1, i_2, \ldots, i_j \geqslant 2$ and $1 \leqslant j \leqslant k$. We also define

$$K = \sup_{t>0} \left\| t S'(t) \right\|.$$

LEMMA 1.1. *If a semigroup S is differentiable and $K < \infty$, then the Euler approximations $E_n(t)$ allow the asymptotic expansion*

$$E_n(t) = S(t) + \frac{a_1}{n} + \cdots + \frac{a_k}{n^k} + r_k, \quad \textit{for } n \geqslant 2, \tag{1.2}$$

with

$$a_m = \sum_{j=m+1}^{2m} c_{m,j-m} S^{(j)}(t) t^j, \tag{1.3}$$

for $m = 1, 2, \ldots$.

The asymptotic expansion (1.2) and the bounds for the remainder terms r_k were obtained by Bentkus (see Theorem 1.3 in [2]) using the Laplace transforms. In this Lemma we obtained expressions (1.3) for the coefficients a_m.

In [10] we obtained another form of coefficients $c_{k,j}$ using an alternative (direct) approach which is not based on Laplace transforms. Also it is easy to obtain the recurrence relations for $c_{k,j}$. From (1.1) we get (this can be easily checked using induction)

$$c_{m,1} = \frac{1}{m+1}, \quad c_{m,j} = \frac{1}{j} \sum_{k=j-1}^{m-1} \frac{c_{k,j-1}}{m-k+1} \tag{1.1a}$$

for $j = 2, \ldots, m$ and $m = 1, 2, \ldots$. From expression (1.10) in [10] we obtain one more recurrence relation

$$c_{m,1} = \frac{1}{m+1}, \quad c_{m,j} = \frac{1}{m+j} \sum_{k=j-1}^{m-1} c_{k,j-1} \tag{1.1b}$$

for $j = 2, \ldots, m$ and $m = 1, 2, \ldots$.

We note that the derivatives $E_n^{(s)}(t)$, $s = 1, 2, \ldots$ allow the asymptotic expansion similar to (1.2). In order to obtain these expansions one can term-wise differentiate (1.2).

Now we provide explicit expressions for asymptotic expansions of the semigroup $S(t)$ in a series of powers of n^{-1} with coefficients b_k depending on derivatives of $E_n(t)$, i.e., asymptotic expansions

$$S(t) = E_n(t) + \frac{b_1}{n} + \ldots + \frac{b_k}{n^k} + \Delta_k, \quad \text{for } n \geqslant 2 \tag{1.4}$$

In order to establish these expansions, we have to establish expansions for the derivatives $S^{(m)}(t)$ as well, $m = 1, 2, \ldots$ Then the coefficients in (1.4) are given by

$$b_0 = E_n(t), \quad b_m = -\sum_{l=1}^{m} \sum_{j=l+1}^{2l} c_{l,j-l} t^j b_{m-l}^{(j)}, \quad m = 1, 2, \ldots, \tag{1.5}$$

where $c_{i,j}$ are given by (1.1). For example, we have

$$b_1 = -\frac{t^2}{2} E_n^{(2)}(t),$$

$$b_2 = \frac{t^2}{2} E_n^{(2)}(t) + \frac{2t^3}{3} E_n^{(3)}(t) + \frac{t^4}{8} E_n^{(4)}(t),$$

$$b_3 = -\frac{t^2}{2} E_n^{(2)}(t) - 2t^3 E_n^{(3)}(t) - \frac{3t^4}{2} E_n^{(4)}(t) - \frac{t^5}{3} E_n^{(5)}(t) - \frac{t^6}{48} E_n^{(6)}(t).$$

We also denote

$$b_0^{(s)} = E_n^{(s)}(t), \quad b_k^{(s)} = -\sum_{l=1}^{k} \sum_{j=l+1}^{2l} c_{l,j-l} \sum_{i=0}^{\min(s,j)} \frac{j!}{(j-i)!} C_s^i t^{j-i} b_{k-l}^{(j+s-i)}, \quad (1.6)$$

for $k = 1, 2, \ldots$ and $s = 0, 1, 2, \ldots$. In case where $s = 0$ we obtain coefficients b_m given by (1.5).

LEMMA 1.2. *If semigroup S is differentiable and $K < \infty$, then the derivatives of $S(t)$ allow the asymptotic expansions*

$$t^s S^{(s)}(t) = t^s E_n^{(s)}(t) + \frac{t^s b_1^{(s)}}{n} + \ldots + \frac{t^s b_k^{(s)}}{n^k} + \Delta_k^{(s)}, \quad (1.7)$$

for $s = 0, 1, 2, \ldots$ and $n \geqslant 2$ (when $s = 0$, we have asymptotic expansion (1.4)). *The coefficients $b_m^{(s)}$ are given by* (1.6).

The bounds for the remainder terms were obtained in Theorem 1.8 in [2].

In case of the semigroups $S(t)$ which satisfy the condition

$$S'(t) = S'(0)S(t), \quad (1.8)$$

we obtain simpler expressions for coefficients b_m. Here $S'(0)$ is the derivative (in some sense) of the semigroup at $t = 0$. For example, if $S(t) = e^{tA}$, $t \geqslant 0$, is strongly continuous semigroup of operators, then $S'(0) = A$ is the infinitesimal generator of the semigroup (see, for example, Chapter II in [7]).

We write

$$h_m = \sum_{i=1}^{m} (-1)^i c_{m,i} \big(S'(0)t\big)^{m+i}, \quad m = 1, 2, \ldots. \quad (1.9)$$

LEMMA 1.3. *If differentiable semigroup S satisfies condition* (1.8) *and $K < \infty$, then it allows the asymptotic expansion* (1.4), *where the coefficients*

$$b_0 = E_n(t), \quad b_m = h_m E_n(t), \quad m = 1, 2, \ldots$$

and the remainder term

$$\Delta_s = -r_s - \sum_{k=0}^{s-1} h_{s-k} \frac{r_k}{n^{s-k}},$$

with h_m given by (1.9).

2. Proofs

Proof of Lemma 1.1. In the proof of Theorem 1.3 in [2] it was demonstrated that coefficients a_m in (1.2) are linear combinations of $t^s S^{(s)}(t)$ with some numerical coefficients $c_{m,s}$ depending only on m and s. Since coefficients $c_{m,s}$ do not depend on concrete semigroup, we can determine them by taking, for example, semigroup $S(t) = \mathrm{e}^t$, $t > 0$. We write

$$(1 - t/n)^{-n} - \mathrm{e}^t = \mathrm{e}^t v_n(t),$$

where $v_n(t) = \mathrm{e}^{-t}(1-t/n)^{-n}-1$. Using expansions $\exp x = \sum_{k=0}^{\infty} x^k/k!$ and $\ln(1+x) = \sum_{k=1}^{\infty}(-1)^{k-1}x^k/k$ we get

$$v_n(t) = \exp\Big\{-t - n\ln\Big(1 - \frac{t}{n}\Big)\Big\} - 1 = \exp\Big\{-t - n\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\Big(\frac{-t}{n}\Big)^k\Big\} - 1$$

$$= \exp\Big\{t\sum_{k=1}^{\infty}\frac{1}{k+1}\frac{t^k}{n^k}\Big\} - 1 = \sum_{j=1}^{\infty}\frac{t^j}{j!}\Big(\sum_{k=1}^{\infty}\frac{1}{k+1}\frac{t^k}{n^k}\Big)^j.$$

Raising to the jth power and changing the order of summation we obtain

$$v_n(t) = \sum_{j=1}^{\infty}\frac{t^j}{j!}\sum_{k=j}^{\infty}\frac{t^k}{n^k}\sum_{i_1+\ldots+i_j=k}\frac{1}{(i_1+1)\ldots(i_j+1)} = \sum_{k=1}^{\infty}\frac{1}{n^k}\sum_{j=1}^{k}t^{k+j}c_{k,j},$$

where $c_{k,j}$ are given by (1.1). Replacing t^s with $S^{(s)}(t)t^s$ we obtain expression (1.3) for a_m.

Proof of Lemma 1.2. We prove the theorem using induction with respect to k. In case when $k = 0$ we have $S(t) = b_0 + \Delta_0$, where $b_0 = E_n(t)$, $\Delta_0 = -r_0$, and $t^s S^{(s)}(t) = t^s b_0^{(s)} + \Delta_0^{(s)}$, for all $s = 1, 2, \ldots$ When $k = 1$, from (1.2) we have

$$S(t) = E_n(t) - \frac{c_{1,1}t^2 S^{(2)}(t)}{n} - r_1.$$

Substituting $t^2 S^{(2)}(t) = t^2 b_0^{(2)} + \Delta_0^{(2)}$ we obtain

$$S(t) = E_n(t) + \frac{b_1}{n} + \Delta_1,$$

where $b_1 = -c_{1,1}t^2 b_0^{(2)}$ and $\Delta_1 = -r_1 - \frac{c_{1,1}}{n}\Delta_0^{(2)}$. Differentiating we get

$$b_1^{(s)} = -c_{1,1}\sum_{i=0}^{\min(s,2)}\frac{2!}{(2-i)!}C_s^i t^{2-i} b_0^{(2+s-i)},$$

for $s = 1, 2, \ldots$.

Assume, that (1.4) and (1.7) hold for $0, 1, \ldots, k-1$ and $s = 1, 2, \ldots$. Let us show that (1.4) and (1.7) hold for k as well. From (1.2) we have

$$S(t) = E_n(t) - \frac{a_1}{n} - \ldots - \frac{a_k}{n^k} - r_k, \tag{2.1}$$

where

$$\frac{a_m}{n^m} = \frac{1}{n^m} \sum_{s=m+1}^{2m} c_{m,s-m} t^s S^{(s)}(t) \tag{2.2}$$

with $c_{m,s}$ given by (1.1). From (1.4) and (1.7) we have

$$S(t) = E_n(t) + \frac{b_1}{n} + \ldots + \frac{b_{k-m}}{n^{k-m}} + \Delta_{k-m},$$

$$t^s S^{(s)}(t) = t^s E_n^{(s)}(t) + \frac{t^s b_1^{(s)}}{n} + \cdots + \frac{t^s b_{k-m}^{(s)}}{n^{k-m}} + \Delta_{k-m}^{(s)} \quad \text{for } s = m+1, \ldots, 2m, \tag{2.3}$$

where $m = 1, 2, \ldots, k-1$. Substituting (2.3) into expression (2.2) we obtain

$$\frac{a_m}{n^m} = \frac{1}{n^m} \sum_{s=m+1}^{2m} c_{m,s-m} t^s \Big(E_n^{(s)}(t) + \frac{b_1^{(s)}}{n} + \ldots + \frac{b_{k-m}^{(s)}}{n^{k-m}} \Big)$$
$$+ \frac{1}{n^m} \sum_{s=m+1}^{2m} c_{m,s-m} \Delta_{k-m}^{(s)}, \tag{2.4}$$

for $m = 1, 2, \ldots, k$. Substituting (2.4) into (2.1), then collecting terms with the same powers of n and moving terms containing the remainder terms into total remainder term Δ_k we obtain expression (1.4) with b_k given by (1.5). Differentiating b_k with respect to t we get expression (1.6) and from here we obtain asymptotic expansion (1.7).

Proof of Lemma 1.3. We first note that if we take in mind the property (1.8), then coefficients a_m in asymptotic expansion (1.2) take the form $a_m = d_m S(t)$, where $d_m = \sum_{j=1}^{m} c_{m,j} (S'(0)t)^{m+j}$, for $m = 1, 2, \ldots$. This means that to obtain the inverse expansion (1.4) we do not need to find the asymptotic expansions of the derivatives of $S(t)$ and $E_n(t)$ as in Lemma 1.2. Using induction on k like in the proof of Lemma 1.2 we then obtain the following recurrence expressions for coefficients b_m in (1.4) :

$$b_0 = E_n(t), \quad b_m = -\sum_{j=1}^{m} d_j b_{m-j}, \quad m = 1, 2, \ldots$$

From here it is easy to obtain another form of these coefficients (this can be checked using induction)

$$b_m = \sum_{r=1}^{m} (-1)^r \sum_{i_1 + \ldots + i_r = m} d_{i_1} \ldots d_{i_r} b_0, \quad m = 1, 2, \ldots.$$

We see that the coefficients b_m have the form $b_m = h_m b_0$, where h_m are the linear combinations of $(S'(0)t)^{m+1}, (S'(0)t)^{m+2}, \dots, (S'(0)t)^{2m}$ with some numerical coefficients which do not depend on concrete semigroup. Therefore, in order to determine them we can take, as in the proof of Lemma 1.1, the semigroup $S(t) = \mathrm{e}^t$. Then we write

$$\mathrm{e}^t - (1 - t/n)^{-n} = (1 - t/n)^{-n} u_n(t),$$

where $u_n(t) = \mathrm{e}^t(1 - t/n)^n - 1$. It's easy to see that $u_n(t) = v_{-n}(-t)$ ($v_n(t)$ was defined in the proof of Lemma 1.1). From here (1.10) follows.

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REZIUMĖ

M. Vilkienė. Pusgrupių Eulerio aproksimacijų asimptotinių skleidinių koeficientų išraiškos

[2] straipsnyje Bentkus pateikė pusgrupių Eulerio aproksimacijų asimptotinius skleidinius. Mes šių skleidinių koeficientus užrašėme išreikštinėje formoje.