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Estimation of parameters for the stochastic linear delay growth law through the L^1 distance procedure

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Abstract. This paper considered deterministic and stochastic delayed linear growth laws for the modeling of the process of growth. For the estimation of parameters the L^1 distance procedure was proposed. As an illustrative experience was used a real data set from repeated measurements on permanent sample plots of pine stands in Lithuania.

Keywords: stochastic delay differential equation, Fokker–Planck equation, stationary density, L^1 distance.

1. Introduction

In ecology, biology and forest biometry literature delay differential equations are widely used to model an oscillatory behavior [1]–[4] of the process of growth. The delay differential equations demonstrated more complicated dynamics than ordinary differential equations. The oscillations in the solutions of deterministic first order delay differential equations are generated by the delayed argument. It is widely recognized that the growth process operates in a highly uncertain environment. However, most available time-delayed growth laws are deterministic, which do not necessarily give a satisfactory deterministic prediction of mean trends. Many scientists in ecology, forest biometry agree that stochastic perturbations are a major determinant of the process of growth. Fluctuations of the growth dynamics can be modeled in various ways. For example, it can be suggested to consider extensive (multiplicative, state-dependent) or intensive (additive, noise amplitude) random perturbations. As was shown [5], the additive and multiplicative noise perturbations of Ito type in a scalar delay deterministic equation may induce the oscillations in the previous deterministic non-oscillatory system. A periodic behavior of the stochastic model is amplified by the combination of noise and delay. The interaction of noise and delay supports oscillations. This phenomenon has been entitled as an autonomous stochastic resonance [1]. It appears that in many ecological systems the growth process exhibit only small fluctuations around steady-state fixed points. This fact is a mathematical motivation to use the linear systems by delays and weak perturbation noises for the modeling of the growth processes in ecology, biology and forestry.

2. Materials and methods

Stochastic growth models have pioneered the use of transition probability densities to describe movements among state levels [6], [7]. This methodology has been rarely adapted due to difficulties with estimating the transition probability density. The transition probability methodology has been based on the assumption that we deal with Markov processes. Unfortunately, in many cases, stochastic growth law with time-delayed feedback is described in terms of a stochastic delay differential equation. So, there is an effect of the long time memory of a growth process on its current behavior and require a description in terms of non-Markov processes. Let us consider a stochastic growth law described by an ordinary delay stochastic differential equation with both the additive and multiplicative noise in the following form

$$\mathrm{d}X(t) = f\big(X(t), X(t-\tau)\big)\,\mathrm{d}t + \sigma \begin{cases} \mathrm{d}W(t), \\ g\big(X(t), X(t-\tau)\big)\,\mathrm{d}W(t), \end{cases} \tag{1}$$
$$X(s) = \varphi(s), \quad s \in [t_0 - \tau; t_0], \quad t \in [t_0; T],$$

when τ, σ are nonnegative real numbers. In general, stochastic processes can be characterized by means of transition probability densities of nonlinear Fokker–Planck equations. Next, we discuss a transition probability density of the stochastic process $X(t)$ described by (1). In the sequel $p(x,t)$ denote the probability density of the stochastic delay differential Eq. 1. It is shown [8], [9], that the delay Fokker–Planck equation of the stochastic process described by (1) takes the form

$$\begin{aligned}\frac{\partial}{\partial t} p(x,t) = &-\frac{\partial}{\partial x} \int f(x, x_\tau) p(x,t; x_\tau, t-\tau)\,\mathrm{d}x_\tau \\ &+ \frac{\sigma^2}{2} \int \frac{\partial^2}{\partial x^2} g(x, x_\tau) p(x,t; x_\tau, t-\tau)\,\mathrm{d}x_\tau,\end{aligned} \tag{2}$$

here $p(x,t; x_\tau, t-\tau)$ is the joint probability density of the stochastic process $X(t)$. The approximate stationary probability density for the stochastic delay differential Eq. 1 that involves the multiplicative noise has the following form [9]

$$p_{st}^{a}(x) = \frac{N_c}{D_{eff}(x)} \exp\Big(\int^{x} \frac{f_{eff}(x')}{D_{eff}(x')}\,\mathrm{d}x'\Big), \tag{3}$$

where N_c is a normalizing constant, the effective drift and diffusion coefficients $f_{eff}(x)$, $D_{eff}(x)$ are described by

$$f_{eff}(x) = \sqrt{\frac{1}{4\pi D^{(0)}(x)\tau}} \int_{-\infty}^{+\infty} f(x, x_\tau) \exp\Big(-\frac{(x_\tau - x - f^{(0)}(x)\tau)^2}{4D^{(0)}(x)\tau}\Big)\,\mathrm{d}x_\tau, \tag{4}$$

$$D_{eff}(x) = \sqrt{\frac{1}{4\pi D^{(0)}(x)\tau}} \int_{-\infty}^{+\infty} \sigma^2 g^2(x, x_\tau) \exp\Big(-\frac{(x_\tau - x - f^{(0)}(x)\tau)^2}{4D^{(0)}(x)\tau}\Big)\,\mathrm{d}x_\tau \tag{5}$$

with $D^{(0)}(x) = \sigma^2 g^2(x,x)/2$, $f^{(0)}(x) = f(x,x)$. For the stochastic delay differential Eq. 1 that involves the additive noise the approximate stationary probability density has the following form

$$p_{st}^{a}(x) = N_c \exp\Big(\frac{2V_{eff}(x)}{\sigma^2}\Big), \tag{6}$$

where the effective potential $V_{eff}(x)$ is described by

$$V_{eff}(x)=\frac{1}{\sigma\sqrt{2\pi\tau}}\int^{x}\Big(\int_{-\infty}^{+\infty}f(x',x_\tau)\exp\Big(-\frac{(x_\tau-x'-f^{(0)}(x')\tau)^2}{2\sigma^2\tau}\Big)\,\mathrm{d}x_\tau\Big)\,\mathrm{d}x'. \quad (7)$$

and the normalizing constant N_c is determined from the normalization condition

$$\int_{-\infty}^{+\infty} p_{st}^{a}(x)\,\mathrm{d}x = 1.$$

Linear delay differential equation

In this section we consider the linear case of (1) with both additive and multiplicative noise

$$\mathrm{d}X(t) = \big(K - aX(t) - bX(t-\tau)\big)\,\mathrm{d}t + \sigma\begin{cases}\mathrm{d}W(t)\\ X(t)\,\mathrm{d}W(t),\end{cases} \quad (8)$$

$$X(s) = \varphi(s), \quad s \in [t_0-\tau; t_0], \quad t \in [t_0; T],$$

where parameters a, b ($a+b>0$) correspond to friction coefficients, $\sigma > 0$ corresponds to the fluctuation strength, $K/(a+b)$ corresponds to the equilibrium fix point, τ the size of delay, the function $\varphi(s)$ describes the initial condition of the stochastic process on $[t_0-\tau; t_0]$, $W(t)$ is a scalar standard Brownian motion (white noise).

Deterministic linear equation

The linear delay differential equation has the form

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = K - ax(t) - bx(t-\tau), \quad (9)$$

$$x(s) = \varphi(s), \quad s \in [t_0-\tau; t_0], \quad t \in [t_0; T],$$

when K, a, b, τ are nonnegative parameters, $a+b>0$. The solutions of (9) are oscillatory provided $b\tau > \frac{1}{e}$. The presence of a delay term in the scalar linear differential (9) induces the oscillation of all solutions if the delay is sufficiently long or large intensity. Nonoscillatory solutions can still exist for small intensity or delay. Finally, the solutions of (9) undergo damped oscillations for $\frac{1}{e} < b\tau < \frac{\pi}{2}$, and diverging oscillations for $b\tau > \frac{\pi}{2}$.

Generally, the parameters K, a, b, τ have to be estimated from past data on the dependent variable x and independent variable t. The problem of estimating parameters K, a, b, τ from historical data is one of choosing estimates $\widehat{K}$, $\hat{a}$, $\hat{b}$, $\hat{\tau}$, such that the predicted value $\hat{x}(t_i)$ is close to observation x_i. An exact expression of the solution of (9) cannot be derived. Before applying the least squares estimates of unknown parameters we need to define a numerical approximation of the solution of (9). The order 2.0 numerical approximation is defined by the relation

$$\begin{aligned} x_{i+1} = x_i &+ (K - ax_i - bx_{i-\theta})\Delta t + (K - ax_i - bx_{i-\theta})\frac{\Delta t^2}{2} \\ &+ I_{\{i>\theta\}}b(bx_{i-2\theta} + ax_{i-\theta} - K)\frac{\Delta t^2}{2}, \end{aligned} \quad (10)$$

here $I_{\{\cdot\}}$ is the indicator function, $\Delta t = \frac{T-t_0}{N}$ is the step size, $\theta = \frac{\tau}{\Delta t}$, $i = -\theta, -\theta + 1, \dots, 0, \dots, N$.

Using Eq. 10, we can define the least squares estimates of unknown parameters.

Stochastic linear equation

In this section for the stochastic linear delay differential Eq. 8 we pose two questions. First, how can be expressed approximate stationary probability and exact stationary probability densities. Second, how can the parameters K, a, b, τ, σ be estimated from experimental data?

There are many approaches to estimating the parameters. Very popular approaches for the estimating of parameters are the maximum likelihood procedure and the L^1 distance procedure [10]. The parameters K, a, b, τ, σ may be divided into two groups: the drift parameters K, a, b, τ and the diffusion parameter σ. The drift parameters may be estimated by the least squares method described in the previous section for the deterministic case. In this paper to estimating of all parameters we will use the L^1 distance procedure.

As suggested in [8], for small delays and diffusion function of the form $g(X(t))$, the drift term $K - aX(t) - bX(t-\tau)$ and diffusion terms σ, $\sigma X(t)$ replacing by $(1+b\tau)(K-(a+b))X(t))$, $(1+b\tau)\sigma$ and $(1+bt)\sigma X(t)$, respectively, the stochastic delay linear differential Eq. 8 takes the approximate non-delay form

$$\mathrm{d}X(t) = (1+b\tau)\big(K-(a+b)X(t)\big)\,\mathrm{d}t + (1+b\tau)\sigma\,\mathrm{d}W(t) \tag{11a}$$

for the additive noise, and

$$\mathrm{d}X(t) = (1+b\tau)\big(K-(a+b)X(t)\big)\,\mathrm{d}t + (1+b\tau)\sigma X(t)\,\mathrm{d}W(t) \tag{11b}$$

for the multiplicative noise.

For the stochastic non-delay linear ($b = 0, \tau = 0$) differential Eq. 8 exists the exact stationary density $p(x)$ of the Fokker–Planck (2). The stationary solution for the additive noise takes the form

$$p_{st}(x) = \frac{1}{\sigma}\sqrt{\frac{a}{\pi}}\exp\left(-\frac{a\left(x-\frac{K}{a}\right)^2}{\sigma^2}\right), \tag{12}$$

and, for the multiplicative noise

$$p_{st}(x) = \frac{1}{\Gamma\left(\frac{2a}{\sigma^2}+1\right)}\left(\frac{2K}{\sigma^2}\right)^{\frac{2a}{\sigma^2}+1} x^{-2\left(\frac{a}{\sigma^2}+1\right)}\exp\left(-2\frac{K}{\sigma^2 x}\right), \tag{13}$$

where $\Gamma(\cdot)$ is a gamma function.

Now, let us study the approximate stationary density of the stochastic linear delay process with the additive noise described by (8). In the linear delay case (8) we have $f(x, x_\tau) = K - ax - bx_\tau$, $f^{(0)}(x) = K - (a+b)x$. So, Eqs. 6–7 give the following forms

$$V_{eff} = (1-b\tau)Kx - (1-b\tau)(a+b)\frac{x^2}{2},$$

$$p^a_{st}(x) = \frac{1}{\sigma}\sqrt{\frac{(1-b\tau)(a+b)}{\pi}}\exp\left(-\frac{(1-b\tau)(a+b)}{\sigma^2}\left(x-\frac{K}{a+b}\right)^2\right). \tag{14}$$

Taking into account that $\frac{1}{1+b\tau} = 1 - b\tau + \mathrm{O}(\tau^2)$, we have

$$p^a_{st}(x) = \frac{1}{\sigma}\sqrt{\frac{a+b}{(1+b\tau)\pi}}\exp\left(-\frac{a+b}{(1+b\tau)\sigma^2}\left(x-\frac{K}{a+b}\right)^2\right). \tag{15}$$

Replacing K by $(1+b\tau)K, a$ by $(1+b\tau)(a+b)$, and $\sigma^2$by $(1+b\tau)^2\sigma^2$ in (12), give us the approximate stationary probability density (15) of the stochastic process $X(t)$described by (8) with the additive noise. We see that these two approaches for the stochastic linear delay with the additive noise give the same approximate stationary probability density.

Next investigate the exact stationary solution of the Fokker–Planck (2) for the stochastic differential Eq. 8 with the additive noise. In the linear additive case (8), the delay Fokker–Planck (2) takes the form

$$\begin{aligned}\frac{\partial}{\partial x}p(x,t) = {} & \frac{\partial}{\partial x}\left((ax-K)p(x,t) + b\int_{-\infty}^{+\infty} yp(x,t;y,t-\tau)\,\mathrm{d}y\right)\\ & + \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2}p(x,t).\end{aligned} \tag{16}$$

Due to the Gaussian distribution of fluctuation force and the linear form of drift function, the exact stationary probability densities $p_{st}(x)$, $p_{st}(x,t;y,t-\tau)$ can been defined by

$$p_{st}(x) = \frac{1}{\sqrt{2\pi R(\tau)}}\exp\left(-\frac{\left(x-\frac{K}{a+b}\right)^2}{2R(\tau)}\right), \tag{17}$$

$$\begin{aligned}p_{st}(x,t;y,t-\tau) = {} & \frac{c(\tau)\sqrt{1-d^2(\tau)}}{\pi}\exp\left(-c(\tau)\left(\left(x-\frac{K}{a+b}\right)^2 + \left(y-\frac{K}{a+b}\right)^2\right.\right.\\ & \left.\left. -2d(\tau)\left(x-\frac{K}{a+b}\right)\left(y-\frac{K}{a+b}\right)\right)\right),\end{aligned} \tag{18}$$

where the parameters $c(\tau), d(\tau)$ and variance $R(\tau)$ are unknown. Substituting (17) and (18) into (16) we have

$$c(\tau) = \frac{2b^2R(\tau)}{(2bR(\tau))^2 - (\sigma^2 - 2aR(\tau))^2},$$

$$d(\tau) = \frac{\sigma^2 - 2aR(\tau)}{2bR(\tau)}.$$

For deriving the variance $R(\tau)$ of the process $X(t)$was used the approach developed in [11]. The variance $R(\tau)$ takes the form

$$R(\tau)=\begin{cases} \frac{\sigma^2}{2}\Big(\frac{1+b\varpi^{-1}\sin(\varpi\tau)}{a+b\cos(\varpi\tau)}\Big), & b>a\geqslant 0,\\ \frac{\sigma^2}{2}\Big(\frac{1+b\varpi^{-1}\sinh(\varpi\tau)}{a+b\cosh(\varpi\tau)}\Big), & a>b\geqslant 0,\\ \frac{\sigma^2}{2}(1+a\tau), & a=b>0, \end{cases} \tag{19}$$

where $\varpi=\sqrt{|a^2-b^2|}$.

For the limitary case ($\tau\to 0$) the exact stationary probability density (17) coincides with the non-delayed stationary probability density (12).

Now we are dealing with the approximate stationary probability density of the stochastic linear delay differential Eq. 8 with the multiplicative noise. Using the probability density approach we have that $f(x,x_\tau)=K-ax-bx_\tau$, $f^{(0)}(x)=K-(a+b)x$, $D^{(0)}(x)=\frac{\sigma^2x^2}{2}$ and Eqs. 3–5 take the following form

$$\begin{aligned} &f_{eff}(x)=(1-b\tau)\big(Kx-(a+b)x\big),\quad D_{eff}(x)=\sigma^2x^2,\\ &p_{st}^{1a}(x)=\Big(\frac{K}{(1-b\tau)\sigma^2}\Big)^{\frac{(a+b)}{(1-b\tau)\sigma^2}}\Big/\Gamma\Big(\frac{(a+b)}{(1-b\tau)\sigma^2}\Big)\\ &\qquad\times x^{-(\frac{(a+b)}{(1-b\tau)\sigma^2}+1)}\exp\Big(-\frac{K}{(1-b\tau)\sigma^2x}\Big). \end{aligned} \tag{20}$$

Second, we may use the approach of the approximation (11b) of stochastic differential equation. Replacing K by$(1+b\tau)K$, a by $(1+b\tau)(a+b)$, and σ^2 by $(1+b\tau)^2\sigma^2$ in (13) and taking into account that $\frac{1}{1+b\tau}=1-b\tau+\mathrm{O}(\tau^2)$, give us the approximate stationary probability density (21)

$$\begin{aligned} p_{st}^{2a}(x)=&\Big(\frac{2(1-b\tau)K}{\sigma^2}\Big)^{\frac{2(1-b\tau)(a+b)}{\sigma^2}+1}\Big/\Gamma\Big(\frac{2(1-b\tau)(a+b)}{\sigma^2}+1\Big)\\ &\times x^{-2(\frac{(1-b\tau)(a+b)}{\sigma^2}+1)}\exp\Big(-2\frac{(1-b\tau)K}{\sigma^2x}\Big). \end{aligned} \tag{21}$$

Now we discuss the minimizing of the L^1 distance between the observed density (histogram) $p_e(x,t)$ and the fitted density (stationary probability density) $p(x;K,a,b,\tau,\sigma)$. The function $p_e(x,t)$ depends on the observed data and the function $p(x;K,a,b,\tau,\sigma)$ depends on the used stochastic growth law. The L^1 distance is defined by

$$d(K,a,b,\tau,\sigma)=\frac{1}{m}\sum_{j=1}^{m}\int_0^{+\infty}\big|p_e(x,t^j)-p(x;K,a,b,\tau,\sigma)\big|\,\mathrm{d}x, \tag{22}$$

where m is the number of division of time. In order to simulate numerically the integral defined by the right-hand side of (22), we define the observed density as

$$p_e(x^i, t^j) = \frac{1}{\Delta \cdot n} \sum_{k=1}^{n} 1_{\{x^i - \Delta/2 \leqslant x_k < x^i + \Delta/2\}},$$

where $1_{\{x^i - \Delta/2 \leqslant x_k < x^i + \Delta/2\}}$ is one if the observation x_k is in $[x^i - \Delta/2; x^i - \Delta/2[$ and zero otherwise, $x^i = \Delta \cdot i$, n is the number of observations, Δ is the step size. Hence, the numerical approximation of (22) takes the form

$$d_1(K, a, b, \tau, \sigma) \approx \frac{1}{m} \sum_{j=1}^{m} \sum_{i=1}^{\infty} \left| p_e(x^i, t^j) - p(x^i; K, a, b, \tau, \sigma) \right| \Delta.$$

3. Illustrative numerical examples

Let us discuss a numerical example to illustrate theory established in the previous section. The height of an individual tree is an important component in long-term forest planning systems. The set consists of 25 sample plots with 1583 measurements of the

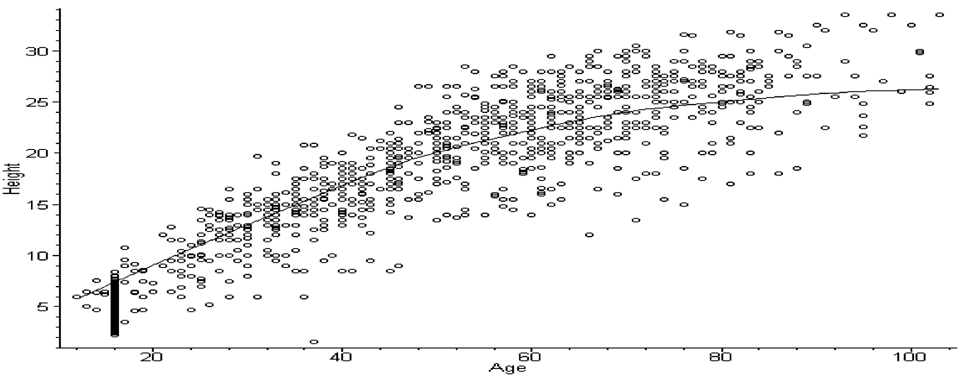

Fig. 1. Plot of the height including data from Pine forests in Lithuania and growth trajectory.

Table 1. L^1 distance estimates of parameters

Density	Parameter					R^2
	K	a	b	τ	σ	
Eq. 12	0.5841	0.0178			2.4713	0.8826
Eq. 13	0.6046	0.01345			0.0981	0.8167
Eq. 15	0.3535	0.0027	0.0118	39.9084	0.5231	0.8845
Eq. 20	0.3717	0.0016	0.0126	50.1181	0.0248	0.8763
Eq. 21	0.3622	0.0040	0.0104	34.3234	0.0207	0.8851

height and age in Pine forests of Lithuania (see Fig. 1). The estimate of parameters when apply the L^1 procedures are presented in Table 1.

An analysis of the results in Table 1 based on the coefficient of determination indicates that all approximate stationary densities suitably fit for the estimating of growth parameters.

References

1. A. Beuter, J. Belair, C. Labrie, Feedback and delays in neurological diseases: a modelling study using dynamical systems, *Bull. Math. Biol.*, **55**, 524–541 (1993).
2. U. Foryš, A. Marciniak-Czochra, Logistic equations in tumour growth modeling, *Int. J. Appl. Math.*, **13**, 317–325 (2003).
3. D. Schley, M.A. Bees, The role of time delays in a non-autonomous host-parasitoid model of slug biocontrol with nematodes, *Ecol. Modell.*, **193**, 543–549 (2006).
4. D. Švitra, V. Denisovas, N. Juščenko, Computer modelling of density dynamics of single-species laboratory insects's population, *Nonlinear Analysis: Modelling and Control*, **9**(4), 327–340 (2004).
5. J.A.D. Appleby, C. Kelly, Oscillation and non-oscillation in solutions of nonlinear stochastic delay differential equations, *Elect. Comm. in Probab.*, **9**, 106–118 (2004).
6. J.H. Matis, T.R. Kiffe, E. Renshaw, J. Hassan, A simple saddlepoint approximation for the equilibrium distribution of the stochastic logistic model of population growth, *Ecol. Model.*, **161**, 139–248 (2003).
7. K. Tanaka, A stochastic model of diameter growth in an even-aged pure forest stand, *J. Jpn. For. Sci.*, **68**, 226–236 (1988).
8. S. Guillouzic, I. L'Heureux, A. Longtin, Small delay approximation stochastic delay differential equations, *Physical Review E*, **59**, 3970–3982 (1999).
9. T.D. Frank, Fokker–Planck perspective on stochastic delay systems: exact solutions and data analysis of biological systems, *Physical Review E*, **68**, 021912 (2003).
10. P. Rupšys, On parameter estimation for stochastic logistic growth laws through the maximum likelihood and L^1 norm procedures, *Lithuanian Statistics*: *Articles, Reports and Studies*, **42**, 49–60 (2005).
11. T.D. Frank, Delay Fokker–Plank equations, Novikov's Theorem, and Boltzmann distributions as small delay approximations, *Physical Review E*, **72**, 011112 (2005).

REZIUMĖ

P. Rupšys. Parametrų įvertinimas L^1 normos metodu stochastiniame tiesiniame su vėlinimu augimo modelyje

Darbe augimo procesui modeliuoti yra naudojamas stochastinis tiesinis su vėlinimu modelis. Augimo proceso parametrų įvertinimui pritaikyta L^1 atstumo procedūra. Rezultatai iliustruojami panaudojant Lietuvos pušies medynų stebėjimo duomenis.