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Forms of anisotropy for spatial variograms *

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Abstract. Frequently an observed spatial variable may have anisotropic variability which varies significantly with direction. In this article overview of anisotropy forms for spatial variograms is presented. Different forms of variograms anisotropy were obtained by fitting salinity data collected at the coastal zone of Baltic Sea. Analysis was carried out using R, a system for statistical computation and graphics.

Keywords: variogram, range, sill, nugget effect, geometric anisotropy, zonal anisotropy.

1. Introduction

Geostatistics is concerned with spatial data. That is, each data value is associated with a location in space and there is at least an implied connection between the location and the data value. The main idea of geostatistical methods is to relate the spatial variation among population densities to the distance lag. Spatially independent data show the same variability regardless of the location of data points. However, spatial data in most cases are not spatially independent. Data values, which are close spatially, show less variability than data values, which are farther away from each other. The exact nature of this pattern varies from data set to data set. This variability is generally computed as a function called semivariance. Semivariogram is very essential in geostatistics. In order to apply kriging to a data set it is necessary to model the variogram.

For spatial locations $\{s_i\colon\ s_i \in D \subset R^d,\ i = \overline{1, N}\}$ in a region D, suppose we observe responses $Z(s_i),\ i = 1, \ldots, N$, where $Z = (Z(s_1), Z(s_2), \ldots, Z(s_N))'$ is viewed as a single observation from a random field. Under the intrinsic hypothesis of Matheron (1963),

$$E(Z(s_1) - Z(s_2)) = 0 \ \text{ and } \ Var(Z(s_1) - Z(s_2)) = 2\gamma(s_1 - s_2) = 2\gamma(h), \qquad (1)$$

where $h = s_1 - s_2$ is separation vector, $2\gamma(h)$ is called the variogram and $\gamma(h)$ is the semivariogram. A stronger assumption is that the process $Z(s)$ is second-order or weakly stationary, (2) (see [2]).

The spatial process $Z(s)$ is said to be isotropic if h is replaced by Euclidean distance, $\|h\| = d$, in (1) and (2) The spatial process is homogeneous if it satisfied both isotropy and second-order stationarity [2].

A typical semivariogram shape is shown in Fig. 1.

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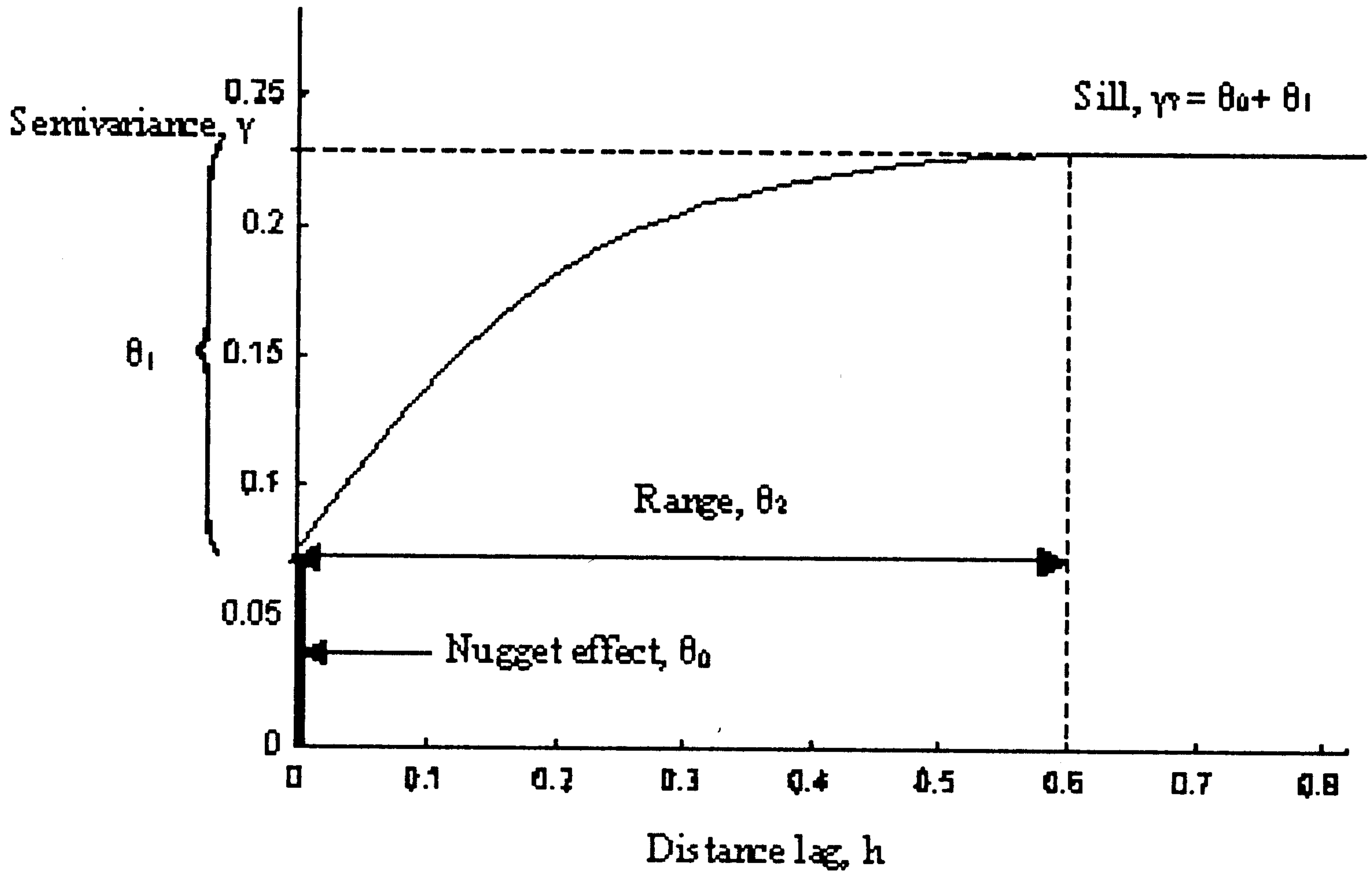

Fig. 1. Semivariogram representation.

Range is the distance at which the semivariogram becomes a constant. Beyond this distance, the mean square deviation between two quantities $Z(s_1)$ and $Z(s_2)$ no longer depends on the distance between them and the two quantities are no longer correlated. If $\lim_{|h|\to\infty} \gamma(|h|) = \gamma_\infty < \infty$, then γ_∞ is called sill of semivariogram. It is the value of the semivariogram for distances beyond its range. Semivariogram may exhibit an apparent discontinuity at the origin. The magnitude of the discontinuity is called the nugget. Nugget effect shows the pure random variation in population density or it may be associated with sampling error. If $\gamma(|h|) \to \theta_0 > 0$ when $|h| \to 0$, then θ_0 is the nugget effect [3].

The spatial process $Z(s)$ is said to exhibit anisotropy when $\gamma(h)$ depends upon both the magnitude and orientation of the separation vector. Anisotropy is usually used to describe a directionally dependent phenomenon.

There is no unified classification of anisotropy forms. Following Zimmerman (1993) anisotropy can take three forms: sill anisotropy, nugget anisotropy and range anisotropy. Range anisotropy can be specified as either geometric (elliptical) range anisotropy or non-geometric range anisotropy. In [2] all anisotropies except geometric is called zonal anisotropy. Other authors mention two forms of anisotropy: geometric anisotropy and zonal anisotropy.

If $\lim_{\delta\to\infty} \gamma(\delta h/||h||)$ depends upon h, we have sill anisotropy (see Fig. 2 a), in other words sills changes with direction. If $\lim_{\delta\to\infty} \gamma(\delta h/||h||)$ depends upon h, we have nugget anisotropy (see Fig. 2 b), i.e., range and sill are constant but nugget changes with direction. With range anisotropy, range changes in different directions (see Fig. 2 c).

In practice geometric anisotropy is most common. Geometric anisotropy, which provides the most common generation of isotropy within stationarity, is typically dealt with by simply transforming the coordinates. Zonal anisotropy can be defined as a nested structure in witch each component of the structure may have its own anisotropy.

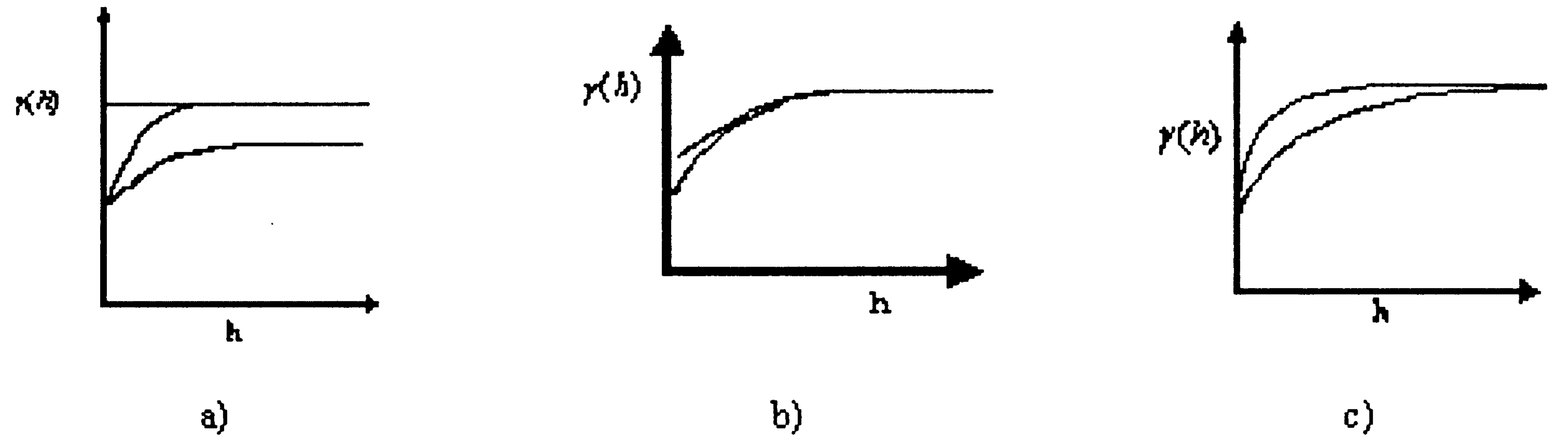

Fig. 2. a) sill anisotropy; b) nugget anisotropy; c) range anisotropy.

A semivariogram or a covariance has a geometric anisotropy when the anisotropy can be reduced to isotropy by a mere linear transformation of the coordinates $\gamma(h_x, h_y) = \gamma'(\sqrt{h_x'^2 + h_y'^2})$, where $\gamma(h_x, h_y)$ is anisotropic semivariance and $\gamma'(\sqrt{h_x'^2 + h_y'^2})$ is isotropic form of semivariance, $h_x' = a_{11}h_x + a_{12}h_y$, $h_y' = a_{21}h_x + a_{22}h_y$, or in matrix or $h' = Ah$, where $A = (a_{ij}), i, j = 1, 2$ represents the matrix of transformation of the coordinates, h and h' are column-vectors of the coordinates.

Consider a geometric (elliptical) anisotropy in two dimensions. Let (x_u, x_v) be the initial rectangular coordinates of a point, φ be the angle made by the major axis of the ellipse with the coordinate axis Ox_u, and $\lambda > 1$ be the ratio of anisotropy of the ellipse. We need to transform this ellipse into a circle and, thus, transform the anisotropy to isotropy.

Given vector (h_u, h_v) , then, to obtain the value of anisotropic semivariogram $\gamma(h) = \gamma(h_u, h_v)$, we first calculate the transformed coordinates (h_u', h_v') from

$$\begin{bmatrix} h_u' \\ h_v' \end{bmatrix} = A \begin{bmatrix} h_u \\ h_v \end{bmatrix}, \quad \text{where } A = \begin{bmatrix} cos^2\varphi + \lambda sin^2\varphi & (1-\lambda) sin\varphi cos\varphi \\ (1-\lambda) sin\varphi cos\varphi & sin^2\varphi + \lambda cos^2\varphi \end{bmatrix},$$

and we substitute these coordinates in the isotropic model, which has a range equal to the major range of the directional ellipse $\gamma(h_u, h_v) = \gamma'(\sqrt{h_u'^2 + h_v'^2})$.

2. Example

The spatial data, used in this article, was collected in the Baltic Sea, where the number of observations is taken once per three months during the period (1994–2001) at 23 stations in the coastal zone. Salinity is the observed feature.

Anisotropy can be detected generating a focused experimental variogram in several different directions and observing whether or not there are significant differences in the resulting variograms. Usually we study angles (φ) of $0°, 45°, 90°, 135°$ with tolerance angle $\epsilon = 45°$. Then each angle group (α_i) is defined as $\varphi - \frac{\epsilon}{2} < a_i < \varphi + \frac{\epsilon}{2}$, $i = 1, 2, 3, 4$.

If anisotropy take place then the variogram of some different directions would be different in either range, sill or nugget.

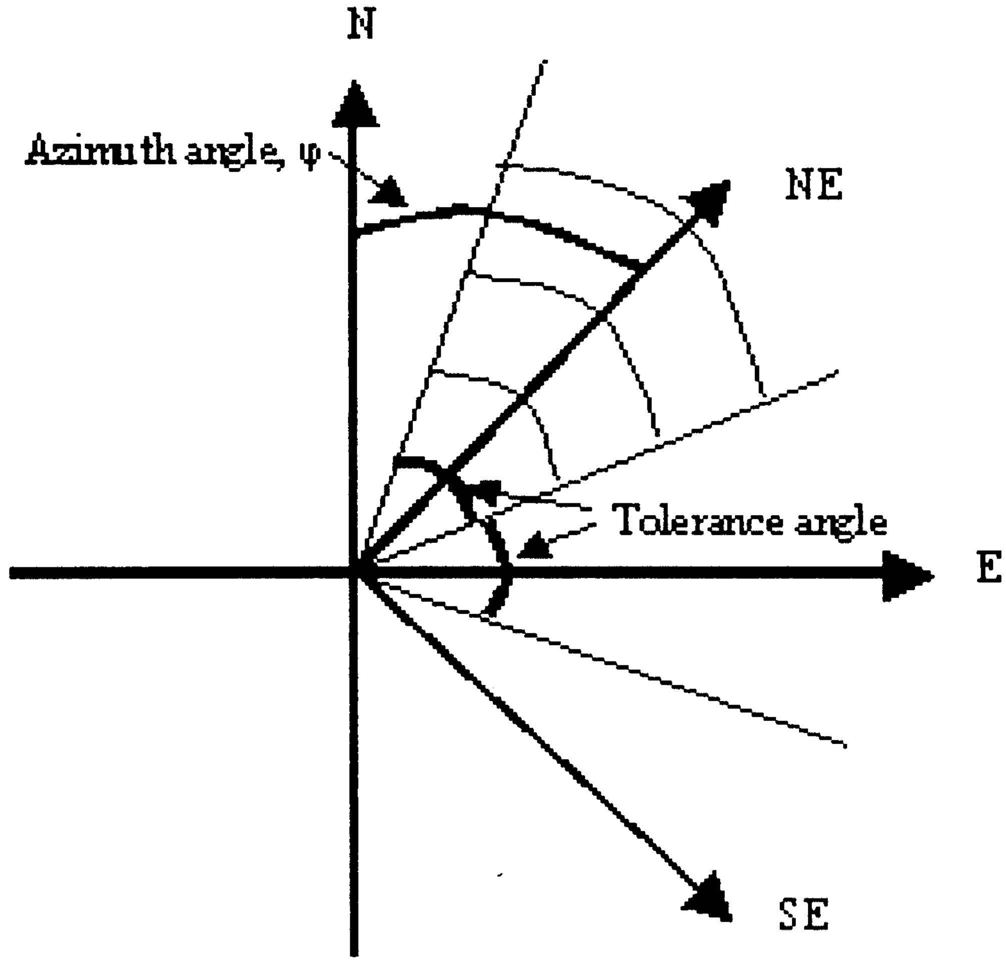

Fig. 3. The directional data used to detect anisotropy.

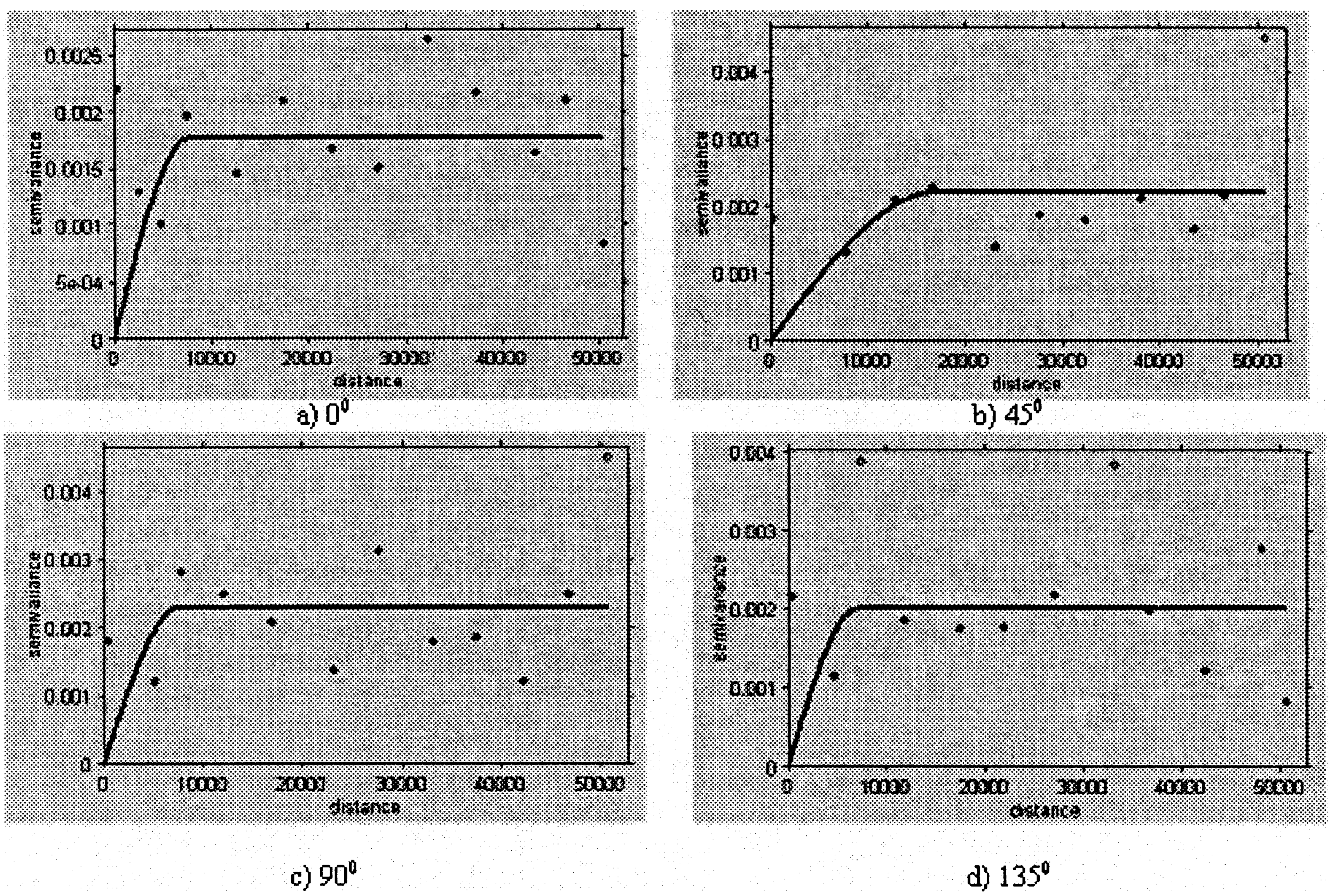

Fig. 4. Directional semivariogram.

Variogram models in four different directions were fitted to collected data. Sills and ranges are presented in Table 1.

Table 1

	Sill	Range
North (0°)	0,001775665	7807,56
Northeast (45°)	**0,002211793**	**17141,55**
East (90°)	0,002299700	7616,75
Southeast (135°)	**0,002002633**	**7078,09**

From table one can conclude that investigated variogram has similar sills in all directions, but range in the Northeast direction significantly differs from the others (see Fig. 4).

So we can conclude that fitted variogram model has range anisotropy with $\varphi = 45°$.

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REZIUMĖ

L. Budrikaitė, K. Dučinskas. Erdvinių variogramų anizotropijos tipai

Šiame straipsnyje metodiškai aprašytos įvairios variogramų anizotropiškumo formos. Šios formos iliustruotos naudojant realius Baltijos jūros druskingumo duomenis bei R sistemos Gstat paketą.