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On the convergence of stochastic processes in the space of discontinuous functions

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We discuss the convergence in distribution of superpositions of integer-valued stochastic processes in the function space $D = D[0,\infty)$ endowed with the Skorohod M_1 topology. As usual, $D[0,\infty)$ denotes the space of all right-continuous functions with left limits, defined on $[0,\infty)$. Stochastic process limits are customarily established by exploiting D endowed with the Skorohod J_1-topology which is stronger than the M_1 topology. However, sometimes the J_1 topology cannot be used, while the M_1 topology works.

The J_1 topology can be introduced by defining the J_1-convergence in D ([1–3]). We say that the sequence $x_n \in D$ J_1-converges to $x \in D$ if there exists a sequence λ_n of continuous strictly increasing functions on $[0,\infty)$ with $\lambda_n(0) = 0$, $\lambda_n(\infty) = \infty$, such that, for each $T > 0$,

$$\lim_{n\to\infty} \sup_{0\leqslant t\leqslant T} \left|\lambda_n(t) - t\right| = 0$$

and

$$\lim_{n\to\infty} \sup_{0\leqslant t\leqslant T} \left|x(\lambda_n(t)) - x_n(t)\right| = 0.$$

The M_1 topology can be introduced in the following way ([1], [4]). For $y \in D[0,T]$ the completed graph of y is the set

$$\Gamma_y = \left\{(z,t) \in R \times [0,T]\colon\ z = \alpha y(t-) + (1-\alpha)y(t) \ \text{ for some } \ \alpha, 0 \leqslant \alpha \leqslant 1\right\},$$

where $y(t-)$ is the left limit of y at t. The order on the completed graphs is defined by saying that $(z_1,t_1) \leqslant (z_2,t_2)$ if either (i) $t_1 < t_2$ or (ii) $t_1 = t_2$ and, $|y(t_1-) - z_1| \leqslant |y(t_2-) - z_2|$. Thus, the order is a total order starting from the "left end" of the completed graph and concluding on the "right end".

A parametric representation of the completed graph Γ_y is a continuous nondecreasing function (u,r) mapping $[0,1]$ onto Γ_y with u being the spatial component and r being the time component. For any $y_1, y_2 \in D[0,T]$ the M_1 metric is defined by

$$d_{\mu_1}(y_1,y_2) = \inf_{\substack{(u_i,r_i)\in\Pi(y_i)\\ i=1,2}} \left\{\|u_1 - u_2\| \vee \|r_1 - r_2\|\right\},$$

where $\Pi(y_i)$ is the set of parametric representations of y_i, $\|\cdot\|$ is the uniform metric, and $a \vee b = \max\{a, b\}$.

We say that the sequence $x_n \in D[0,\infty)$ M_1-converges to $x \in D$ if for each $T > 0$ at which x is continuous, the restrictions of x_n to the subinterval $[0, T]$ converge in the space $D[0, T]$ to the restriction of x with respect to the M_1 metric.

By the integer-valued stochastic process we mean the process $X(t)$, $t \geqslant 0$, for which increments $X(t) - X(s)$, $s < t$, are integer-valued random variables, and $X(0) = 0$ a.s. If the values $X(t) - X(s)$ are non-negative, $X(t)$ is called a point process.

For $\Delta = (s, t]$, let $X(\Delta) = X(t) - X(s)$.

The integer-valued process $X(t)$ with independent increments will be called a compound Poisson process if for each Δ

$$\mathrm{E}\exp\{i\,uX(\Delta)\} = \exp\Big\{\sum_{\substack{m\neq 0\\ m\in Z}} \lambda_m(\Delta)\big(e^{ium} - 1\big)\Big\}, \tag{1}$$

where $\lambda_m(\cdot)$ is a measure ($m \neq 0$) and $\sum_{m\neq 0} |m|\lambda_m(\Delta) < \infty$ ([5], [6]). If $\lambda_1(\cdot) = \Lambda(\cdot)$, $\lambda_m(\cdot) \equiv 0$ for $m \neq 1$, $X(t)$ is the Poisson process with $\mathrm{E}X(t) = \Lambda(t)$.

We denote by $X_n \overset{d,J_1}{\to} X$ or $X_n \overset{d,M_1}{\to} X$ the convergence in distribution of the sequence of stochastic processes X_n to X in the space D endowed with the J_1 or M_1 topology, respectively.

Let

$$\left\{\begin{aligned} &X_{nk}(t), k = 1,\dots,k_n,\ n \geqslant 1,\\ &\lim_{n\to\infty} \max_{1\leqslant k\leqslant k_n} \mathrm{P}\{X_{nk}(t) > 0\} = \lim_{n\to\infty} \sum_{k=1}^{k_n} \mathrm{P}\{X_{nk}(t) > 1\} = 0,\ t > 0 \end{aligned}\right\} \tag{2}$$

be infinitesimal array of independent point processes. Denote

$$X_n(t) = \sum_{k=1}^{k_n} X_{nk}(t), \tag{3}$$

$$\Lambda_n(t) = \sum_{k=1}^{k_n} \mathrm{P}\{X_{nk}(t) > 0\}. \tag{4}$$

Let $X_\Lambda(t)$ be the Poisson process with mean $\mathrm{E}X(t) = \Lambda(t)$. Then $X_n \overset{d,J_1}{\to} X_\Lambda$ if and only if Λ_n J_1-converges to Λ (see, e.g., [3], [7]).

An analogous statement holds in the space D with the M_1 topology.

THEOREM 1. *Let X_n be defined by* (3) *and X_Λ be the Poisson process with mean Λ. Then $X_n \overset{d,M_1}{\to} X_\Lambda$ if and only if Λ_n M_1-converges to Λ.*

Proof. By Corollary 12.5.1 in [4] and Theorem 1 in [8], the M_1-convergence of Λ_n to Λ is equivalent to the weak convergence of the distributions of $(X_n(t_1), \dots, X_n(t_m))$ in R^m to those of $(X_\Lambda(t_1), \dots, X_\Lambda(t_m))$ for all continuity points $t_1, \dots, t_m$ of $\Lambda(t)$, which in turn is tantamount to the convergence in distribution of X_n to X_Λ in D with the M_1-topology (cf. [9]).

Let an integer-valued random variable ξ_{nk}^i be attached to the i-th point of the point process $X_{nk}(t)$ in (2), $p_{nk}(m) = \mathrm{P}\{\xi_{nk}^1 = m\}$ and

$$Z_{nk}(t) = \sum_{i \geqslant 1} \xi_{nk}^i \chi\{\tau_{nk}^i \leqslant t\},$$

where τ_{nk}^i is the i-th point of $X_{nk}(t)$, and χ denotes the indicator function. Denote $Z_n(t) = \sum_{k=1}^{k_n} Z_{nk}(t)$. Let $\Lambda(t)$ be a non-decreasing function on $[0, \infty)$, and let C_Λ be the set of continuity points of Λ.

THEOREM 2. *Suppose that*

$$\lim_{n\to\infty} p_{nk}(m) = p(m) \ \text{ and } \sum_{m\geqslant 1} p(m) > 0, \qquad \sum_{m\leqslant -1} p(m) > 0.$$

Then the sequence Z_n converges in distribution in the function space D with the M_1 topology to the compound Poisson process X defined by (1) *with $\lambda_m = p_m\Lambda$, if and only if for each $t \in C_\Lambda$*

$$\lim_{n\to\infty} \Lambda_n(t) = \Lambda(t), \tag{5}$$

and for each $u \notin C_\Lambda$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \int_{(u-\delta, u+\delta)} \left[\Lambda_n(u+\delta) - \Lambda_n(s)\right] \mathrm{d}\Lambda_n(s) = 0. \tag{6}$$

Proof. By Theorem 1 in [3], (5) is the necessary and sufficient condition for the weak convergence of the finite-dimensional distributions of Z_n to those of the compound Poisson process at continuity points of Λ. The tightness of the distributions of Z_n in D can be characterized in terms of the following oscillation function introduced in [4].

For $x \in D$, $t > 0$, and $\delta > 0$, let

$$w(x, t, \delta) = \sup_{t-\delta \leqslant t_1 < t_2 < t_3 \leqslant t+\delta} \left\{ \left\| x(t_2) - [x(t_1), x(t_3)] \right\| \right\},$$

where $\|z - A\|$ is the distance between the point z and subset A in R, and $[a, b]$ is the standard segment.

The sequence of the distributions of Z_n in D with the M_1 topology is tight if and only if for each $T \in C_\Lambda$

$$\lim_{\delta\to 0} \limsup_{n\to\infty} \mathrm{P}\left\{ \sup_{0\leqslant t\leqslant T} w(Z_n, t, \delta) \geqslant 1 \right\} = 0. \tag{7}$$

Thus, we have to prove that (6) and (7) are equivalent. Let $u \notin C_\Lambda$, $T \in C_\Lambda$, and

$$A_n^\delta = \bigcup_{k \neq l} \Big\{ u - \delta < \tau_{nk}^1 < \tau_{nl}^1 \leqslant u + \delta, \xi_{nk}^1 > 0, \xi_{nl}^1 < 0, \tau_{ni}^1 > T \ \text{ for } \ i \neq k, l \Big\}.$$

Obviously,

$$\mathrm{P}(A_n^\delta) \geqslant \mathrm{P}\big\{X_n(T) = 0\big\} \sum_{k \neq l} \int_{(u-\delta, u+\delta)} \Big[\mathrm{P}\big\{X_{nk}(u+\delta) > 0\big\}$$

$$- \mathrm{P}\big\{X_{nk}(s) > 0\big\} \Big] \mathrm{dP}\big\{X_{nl}(s) > 0\big\} \mathrm{P}\big\{\xi_{nk}^1 > 0\big\} \mathrm{P}\big\{\xi_{nl}^1 < 0\big\}.$$

Since $\mathrm{P}(A_n^\delta) \leqslant \mathrm{P}\{\sup_{0 \leqslant t \leqslant T} w(Z_n, t, \delta) \geqslant 1\}$, (7) implies (6).

To prove the converse, put

$$w'(x, t, \delta) = \sup_{t-\delta \leqslant t_1 < t < t_2 < t+\delta} \Big\{ \big|x(t_1) - x(t)\big| \wedge \big|x(t_2) - x(t)\big| \Big\},$$

where $a \wedge b = \min\{a, b\}$. It is obvious that

$$\mathrm{P}\Big\{ \sup_{0 \leqslant t \leqslant T} w(Z_n, t, \delta) \geqslant 1 \Big\} \leqslant \mathrm{P}\Big\{ \sup_{0 \leqslant t \leqslant T} w'(X_n, t, \delta) \geqslant 1 \Big\}. \tag{8}$$

By Theorem 7 in [3], (6) is equivalent to the tightness of distributions of X_n with respect to the J_1 topology. Consequently, (6) implies the relation

$$\lim_{\delta \to 0} \limsup_{n \to \infty} \mathrm{P}\Big\{ \sup_{0 \leqslant t \leqslant T} w'(X_n, t, \delta) \geqslant 1 \Big\} = 0,$$

which, by (8), implies (7).

As mentioned above, (6) characterizes tightness of distributions of X_n and thus (5) and (6) are necessary and sufficient for the J_1-convergence of Λ_n to Λ. So we can formulate the statement in Theorem 2 as follows.

THEOREM 3. *Under the assumption in Theorem 2, Z_n M_1 converges converge to the compound Poisson process X if and only if the sequence Λ_n J_1 converges to Λ.*

Note that the condition of the theorem is not necessary without the assumption on the distributions $p_{nk}(m)$.

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REZIUMĖ

R. Banys. Apie atistiktinių procesų konvergavimą trūkiųjų funkcijų erdvėje

Nagrinėjamas sveikareikšmių atsitiktinių procesų sumų konvergavimas erdvėje $D[0, \infty)$ su Skorochodo topologijomis J_1 ir M_1. Gautos tokių sumų konvergavimo į apibendrintus Puasono procesus sąlygos.