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On the uniform distribution of endomorphisms of s-dimensional torus, II

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This article is a continuation of the article [3]. The object of our investigation is a uniform distribution of endomorphisms of s-dimensional torus Ω_s. Here we prove that the uniform distribution is attained under different conditions than in [3].

Let $\Omega = \Omega_s$ ($s \geqslant 2$) be a s-dimensional torus, and its endomophisms $T:\Omega_s \to \Omega_s$ are defined by

$$T\vec{x} = \vec{x}\,W \pmod 1$$

where W is a non-singular matrix with integer elements, and $\vec{x}$ is a point of a parametric curve

$$\vec{\xi} = \vec{\xi}\big(\varphi_1(t), \dots, \varphi_s(t)\big), \quad a \leqslant t \leqslant b. \tag{1}$$

Let the functions $\varphi_i(t)$ satisfy the condition

$$\big|\varphi_i'''(t)\big| > \gamma_0 > 0, \quad i = 1, \dots, s. \tag{2}$$

In what follows $c_1, c_2, \dots$ are positive constants, $\|a\|$ denotes the distance of a to the closest integer.

Theorem. *Let the characteristic polynom of the matrix W have distinct integer roots $\theta_1, \dots, \theta_s$. Let there exist sufficiently small $c > 0$ and sufficiently large $\omega > 0$ for which the inequality*

$$\big\|m_1\varphi_1(t) + \dots + m_s\varphi_s(t)\big\| \geqslant \frac{c}{(\overline{m}_1 \cdots \overline{m}_s)^{\omega}}, \quad a \leqslant t \leqslant b, \tag{3}$$

holds. Here the non-zero vector $\vec{m} = (\overline{m}_1, \dots, \overline{m}_s)$ has integer coordinates,

$$\overline{m}_k = \begin{cases} 1, & \text{if } m_k = 0, \\ |m_k|, & \text{if } m_k \neq 0. \end{cases}$$

Then the sequence $\{\vec{\xi}\,W^k\}$, $k = 1, 2, \dots$, is uniformly distributed in the torus Ω_s for almost all $t \in [a, b]$.

Proof. We make use of the expansion

$$\vec{\xi} W^m = \sum_{k=1}^{s} L_k(\varphi_1(t), \dots, \varphi_s(t)) \theta_k^m \vec{w}_k, \tag{4}$$

where $\vec{w}_k$ is the eigenvector corresponding to the eigenvalue θ_k and L_k is the linear form of the variables $z_1, \dots, z_s$:

$$L_k(z_1, \dots, z_s) = \sum_{l=1}^{s} v_{kl} z_l, \quad k = 1, \dots, s,$$

where v_{kj} $(k, l = 1, \dots, s)$ are real numbers defined by the matrix W and independent of m. We get from (4)

$$\vec{m} \cdot \vec{\xi} W^m = \sum_{k=1}^{s} L_k(\varphi_1(t), \dots, \varphi_s(t)) (\vec{m} \cdot \vec{w}_k) \theta_k^m. \tag{5}$$

Since at least one product $\vec{m} \cdot \vec{w}_k$ is not zero, suppose that $\vec{m} \cdot \vec{w}_1 \neq 0$. Assume that $|\theta_1| > |\theta_2|$. In this case the numbers $\vec{m} \cdot \vec{w}_1$ and v_{kl} are rational, and $v_{1k} \neq 0$ for some k. Let R be the least common denominator of fractions v_{1k} and $\vec{m} \cdot \vec{w}_1$. Denote

$$\varphi(t) = \sum_{k=1}^{s} R v_{1k} (\vec{m} \cdot \vec{w}_1) \cdot \varphi_k(t) = \sum_{k=1}^{s} n_k \cdot \varphi_k(t),$$

where $n_k = R v_{1k} \cdot (\vec{m} \cdot \vec{w}_1)$ are integers. Then we get from (3) that

$$\|\varphi(t)\| \geqslant \frac{c}{(\overline{n}_1 \cdots \overline{n}_s)^{\omega}} \geqslant \frac{c_1}{(\overline{m}_1 \cdots \overline{m}_s)^{s\omega}}, \tag{6}$$

where c_1 does not depend on $\vec{m}$. The equality (5) and inequality (6) imply now that

$$\vec{m} \cdot \vec{\xi} W^m = \theta_1^m \frac{\varphi(t)}{R} \left(1 + \mathrm{O}\left(\left(\frac{\theta_2}{\theta_1}\right)^m \cdot (\overline{m}_1 \cdots \overline{m}_s)^{s\omega+1}\right)\right). \tag{7}$$

The vector $\vec{m}$ can be chosen so that

$$(\overline{m}_1 \cdots \overline{m}_s)^{s\omega+1} \leqslant \left|\frac{\theta_1}{\theta_2}\right|^{m/2} = \mathrm{e}^{c_2 m}, \quad c_2 > 0, \tag{8}$$

hold. Then we obtain from (7)

$$I_m = \int_a^b \exp(2\pi i \vec{m} \cdot \vec{\xi} W^m)\, \mathrm{d}t = \int_a^b \exp\left(\frac{2\pi i}{R} \theta_1^m \varphi(t)(1 + \mathrm{O}(\mathrm{e}^{-c_2 m}))\right) \mathrm{d}t. \tag{9}$$

Assume that $|\varphi'(t)| \geqslant c_3 > 0$, otherwise the calculation is analogous to [1]. After the change of variables in (9) and some calculation we obtain the inequality

$$|I_m| \leqslant c_4 \mathrm{e}^{-c_3 m}. \tag{10}$$

Consider a function $g(\cdot)$ belonging to the Korobov class of functions $E_s^\alpha(c)$, $\alpha > 2$ (see [1]). Then this function can be expressed in the following way:

$$g(\vec{x}) = \int_{\Omega_s}\!\!\cdots\!\int g(\vec{x})\,\mathrm{d}\vec{x} + \sum_{m_1,\ldots,m_s=-\infty}^{\infty}{}^{*} c(m_1,\ldots,m_s)\exp(2\pi i\vec{m}\cdot\vec{x})$$

where (*) denotes the summation over s-tuples $(m_1,\ldots,m_s)$ for which at least one of $m_1,\ldots,m_s \neq 0$. This implies that

$$\begin{aligned}
&\left|\int_a^b g(\vec{\xi}W^m)\,\mathrm{d}t - (b-a)\int_{\Omega_s}\!\!\cdots\!\int g(\vec{x})\,\mathrm{d}\vec{x}\right| \\
&\quad\leqslant c_5 \sum_{\overline{m}_1,\ldots,\overline{m}_s > A} (\overline{m}_1\cdots\overline{m}_s)^{-\alpha} \\
&\quad\quad + c_6 \sum_{\overline{m}_1,\ldots,\overline{m}_s \leqslant A} (\overline{m}_1\cdots\overline{m}_s)^{-\alpha}\left|\int_a^b \exp\bigl(2\pi i\vec{m}\cdot\vec{\xi}W^m\bigr)\,\mathrm{d}t\right|
\end{aligned}$$

where $A = |\theta_1/\theta_2|^{c_3 m}$.

The first sum on the right-hand side is evaluated using the corresponding lemma in [1]. We obtain

$$\sum_{\overline{m}_1,\ldots,\overline{m}_s > A} (\overline{m}_1\cdots\overline{m}_s)^{-\alpha} < c_7 \ln^2 m\, \mathrm{e}^{-c_3 m}.$$

From (10) we get that the second sum in the same expression does not exceed

$$c_4 \mathrm{e}^{-c_3 m} \sum_{\overline{m}_1,\ldots,\overline{m}_s \leqslant A} (\overline{m}_1\cdots\overline{m}_s)^{-\alpha} \leqslant c_8 \mathrm{e}^{-c_3 m}$$

These estimates imply

$$\int_a^b g(\vec{\xi}W^m)\,\mathrm{d}t = (b-a)\int_{\Omega_s}\!\!\cdots\!\int g(\vec{x})\,\mathrm{d}\vec{x} + c_9 \ln^2 m\,\mathrm{e}^{-c_3 m}.$$

The rest of the proof is analogous to the proof the theorem in [3].

References

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Apie s-mačio toro emdomorfizmų tolygų pasiskirstymą, II

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Straipsnis yra [3] darbo tęsinys. Įrodoma, kad tolygus pasiskirstymas galioja esant skirtingoms, negu minėtame straipsnyje sąlygoms. Parametrinei kreivei, nusakytai (1) formule ir funkcijomis $\varphi_i(t)$, $i = 1, \dots, s$, tenkinančiomis (2) sąlygą, įrodoma

Teorema. *Tegu matricos W charakteristinis polinomas turi sveikąsias šaknis $\theta_1, \dots, \theta_s$. Jei egzistuoja toks pakankamai mažas skaičius $c > 0$ ir pakankamai didelis skaičius $\omega > 0$, kuriems galioja* (3) *nelygybė, tai seka $\{\xi W^m\}$ yra tolygiai pasiskirsčiusi tore Ω_s beveik visiems $t \in [a, b]$.*