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On the zeros of a new zeta-function

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1. Introduction

As usual, let $s = \sigma + it$ be a complex variable. Further, let k and ℓ be positive integers such that k and 4ℓ are coprime. We write $f(x) = \mathrm{O}(g(x))$ and $f(x) \ll g(x)$, resp., when the estimate $|f(x)| \leqslant Cg(x)$ holds for all large x and some absolute constant C. Finally, we define by $r(n)$ the number of representations of a positive integer n as a sum of two integer squares. Then we consider the following Dirichlet series

$$\mathcal{R}\left(s; \frac{k}{4\ell}\right) = \sum_{n=1}^{\infty} \frac{r(n)}{n^s} \exp\left(2\pi i \frac{nk}{4\ell}\right). \tag{1}$$

The function $\mathcal{R}\left(s; \frac{k}{4\ell}\right)$ was introduced in [4],where a truncated Voronoï-type formula for the twisted Möbius transform

$$\sum_{n \leqslant x} r(n) \exp\left(2\pi i \frac{nk}{4\ell}\right)$$

was proved. In this paper we continue to study the properties of the Dirichlet series $\mathcal{R}\left(s; \frac{k}{4\ell}\right)$ and present some results on the zero distribution.

It is well-known that

$$r(n) = 4 \sum_{d|n} \chi(d),$$

where

$$\chi(d) = \begin{cases} (-1)^{\frac{d-1}{2}} & \text{if} \quad d \equiv 1 \bmod 2, \\ 0 & \text{if} \quad d \equiv 0 \bmod 2. \end{cases}$$

χ is the non-principal character modulo 4 (and thus completely multiplicative). Hence, we obtain

$$r(n) \leqslant 4d(n) \ll n^{\varepsilon}, \tag{2}$$

where $d(n)$ is the divisor function and ε denotes an arbitrarily small positive number. Consequently, the series (1) converges absolutely in the half-plane $\sigma > 1$. In [4] it was proved that the function $\mathcal{R}\left(s; \frac{k}{4\ell}\right)$ has an analytic continuation throughout the complex plane except for a simple pole at $s = 1$, and that it satisfies the functional equation

$$\mathcal{R}\left(s; \frac{k}{4\ell}\right) = \frac{\chi(k^*)}{\pi}\left(\frac{\pi}{2\ell}\right)^{2s-1}\Gamma(1-s)^2 \times \times\left(\mathcal{R}\left(1-s; \frac{k^*}{4\ell}\right) - \cos(\pi s)\mathcal{R}\left(1-s; \frac{-k^*}{4\ell}\right)\right), \tag{3}$$

where k^* is given by $kk^* \equiv 1 \bmod 4\ell$. This functional equation is very similar to the one for the Estermann zeta-function, and, as we shall show in the sequel, also its zero distribution is comparable to the one of the Estermann zeta-function (for which we refer to [3]).

2. Zero distribution

Denote the zeros of $\mathcal{R}\left(s; \frac{k}{4\ell}\right)$ by $\rho = \beta + i\gamma$. In view of (2) we find for sufficiently large σ

$$\left|\mathcal{R}\left(s; \frac{k}{4\ell}\right) - 4\exp\left(2\pi i\frac{k}{4\ell}\right)\right| \leqslant \sum_{n=2}^{\infty}\frac{r(n)}{n^\sigma} \leqslant \sum_{n=2}^{\infty}\frac{4d(n)}{n^\sigma} \ll \int_1^\infty x^{\varepsilon-\sigma}\mathrm{d}x < \frac{1}{\sigma-(1+\varepsilon)}.$$

Hence, as $\sigma \to \infty$,

$$\mathcal{R}\left(s; \frac{k}{4\ell}\right) = 4\exp\left(2\pi i\frac{k}{4\ell}\right) + \mathrm{O}\left(\frac{1}{\sigma}\right). \tag{4}$$

Consequently, there exists a positive constant C such that

$$\mathcal{R}\left(s; \frac{k}{4\ell}\right) \neq 0 \quad \text{for} \quad \sigma > C. \tag{5}$$

Notice that C can be estimated explicitly by elementary means; for instance, the rather trivial estimate $d(n) \leqslant n$ leads to $C = 3$. By the functional equation (3) and the non-vanishing of the Gamma-function, $\mathcal{R}\left(s; \frac{k}{4\ell}\right)$ vanishes if and only if

$$\mathcal{R}\left(1-s; \frac{k^*}{4\ell}\right) = \cos(\pi s)\mathcal{R}\left(1-s; \frac{-k^*}{4\ell}\right).$$

Therefore, with the estimate (4) and the zero-free region (5), it follows that for $\sigma < 1-C$ the function $\mathcal{R}\left(s; \frac{k}{4\ell}\right)$ can only have zeros close to the negative real axis. We call zeros ρ of $\mathcal{R}\left(s; \frac{k}{4\ell}\right)$ with $\beta < 1 - C$ *trivial*. In [4] it was shown that for any positive integer n

$$\mathcal{R}\left(1-n; \frac{k}{4\ell}\right) = 0.$$

We call other zeros of $E(s;\frac{k}{l},\alpha)$ *nontrivial*. By the above and the zero-free region (5) the nontrivial zeros lie in the vertical strip

$$1-C\leqslant\sigma\leqslant C. \tag{6}$$

Applying ideas of Littlewood [2] and Levinson and Montgomery [1] we will prove

Theorem 1. *Let $B>C+1$ be a constant. Then, as $T\to\infty$,*

$$\sum_{\substack{\beta>-B\\|\gamma|\leqslant T}}(B+\beta)\;=\;(2B+1)\,\frac{T}{\pi}\log\frac{2T\ell}{\pi e}+\mathrm{O}(\log T).$$

Denote by $N(T;\frac{k}{4\ell})$ the number of nontrivial zeros ρ of $\mathcal{R}\left(s;\frac{k}{4\ell}\right)$ with $|\gamma|\leqslant T$ (according multiplicities). Using the formula of Theorem 1 with $B+1$ instead of B, we get after subtracting the resulting formula from the one above

COROLLARY 1. As $T\to\infty$,

$$N\left(T;\frac{k}{4\ell}\right)=\frac{2T}{\pi}\log\frac{2T\ell}{\pi e}+\mathrm{O}(\log T).$$

Note that the main term in the asymptotic formula does not depend on k.

Multiplying the formula of Corollary 1 with B and subtracting it from the formula of Theorem 1 gives

COROLLARY 2. We have, as $T\to\infty$,

$$\frac{1}{N(T;\frac{k}{4\ell})}\sum_{\substack{\rho\text{ nontrivial}\\|\gamma|\leqslant T}}\beta=\frac{1}{2}+\mathrm{O}(T^{-1}).$$

One may interpret the last formula in the sense that the mean value of the real parts of the nontrivial zeros of $\mathcal{R}\left(s;\frac{k}{4\ell}\right)$ is $\frac{1}{2}$.

3. Proof of Theorem 1

The proof relies on

Lemma 4 (Littlewood). *Let $f(s)$ be regular in and upon the boundary of the rectangle $\mathcal{R}$ with vertices $b, b+iT, c+iT, c$, and not zero on $\sigma=b$. Denote by $\nu(\sigma,T)$ the number of zeros $\rho=\beta+i\gamma$ of $f(s)$ inside the rectangle with $\beta>\sigma$ including those with $\gamma=T$ but not $\gamma=0$. Then*

$$\int_{\mathcal{R}}\log f(s)\,\mathrm{d}s=-2\pi i\int_b^c\nu(\sigma,T)\,\mathrm{d}\sigma.$$

This is an integrated version of the principle of the argument (the proof can be found in [5], §9.9 or [2]).

Let $A = C + 2$. By the condition on B, all nontrivial zeros of $\mathcal{R}\left(s; \frac{k}{4\ell}\right)$ have real parts in $(-B, A)$. Denote by $N(\sigma, T; \frac{k}{l}, \alpha)$ the number of nontrivial zeros ρ of $\mathcal{R}\left(s; \frac{k}{4\ell}\right)$ with $\beta > \sigma$ and $|\gamma| \leqslant T$. Then Littlewood's lemma 4, applied to

$$\mathcal{R}\left(s; \frac{k}{4\ell}\right)(s-1)$$

and the rectangle $\mathcal{L}$ with vertices $A \pm iT$, $-B \pm iT$, gives us

$$\int_{\mathcal{L}} \log \mathcal{R}\left(s; \frac{k}{4\ell}\right) \mathrm{d}s = -2\pi i \int_{-B}^{A} N\left(\sigma, T; \frac{k}{l}, \alpha\right) \mathrm{d}\sigma + \mathrm{O}(1);$$

here the error term occurs from the removed pole at $s = 1$. Therefore,

$$\begin{aligned} &2\pi \sum_{\substack{\beta > -B \\ |\gamma| \leqslant T}} (B + \beta) + \mathrm{O}(1) \\ &= \int_{-T}^{T} \log \left| \mathcal{R}\left(-B + it; \frac{k}{4\ell}\right) \right| \mathrm{d}t - \int_{-T}^{T} \log \left| \mathcal{R}\left(A + it; \frac{k}{4\ell}\right) \right| \mathrm{d}t \\ &\quad - \int_{-B}^{A} \arg \mathcal{R}\left(\sigma - iT; \frac{k}{4\ell}\right) \mathrm{d}\sigma + \int_{-B}^{A} \arg \mathcal{R}\left(\sigma + iT; \frac{k}{4\ell}\right) \mathrm{d}\sigma \\ &=: \sum_{j=1}^{4} I_j. \end{aligned}$$

To define $\log \mathcal{R}\left(s; \frac{k}{4\ell}\right)$ we choose the principal branch of the logarithm on the real axis, as $\sigma \to \infty$; for other points s the value of the logarithm is obtained by analytic continuation.

By the functional equation (3) we have

$$\begin{aligned} &\log \left| \mathcal{R}\left(-B + it; \frac{k}{4\ell}\right) \right| \\ &= -\log \pi - (2B + 1) \log \frac{\pi}{2\ell} + 2 \log |\Gamma(B + 1 - it)| \\ &\quad + \log \left| \mathcal{R}\left(1 + B - it; \frac{k^*}{4\ell}\right) - \cos(-\pi B + \pi i t)\, \mathcal{R}\left(1 + B - it; \frac{-k^*}{4\ell}\right) \right|. \end{aligned}$$

Using Stirling's formula, we obtain for $|t| > 1$

$$\log |\Gamma(B + 1 - it)| = \left(\frac{1}{2} + B\right) \log |t| - \frac{\pi}{2}|t| + \frac{1}{2} \log 2\pi + \mathrm{O}\left(|t|^{-1}\right).$$

Further, by (4) we get for $|t| > 1$

$$\log \left| \mathcal{R}\left(1 + B - it; \frac{k^*}{4\ell}\right) - \cos(-\pi B + \pi i t)\, \mathcal{R}\left(1 + B - it; \frac{-k^*}{4\ell}\right) \right|$$

$$= \log\left|\mathcal{R}\left(1+B-it;\frac{-k^*}{4\ell}\right)\right| + \pi|t| - \log 2 + \mathrm{O}(\exp(-\pi|t|)).$$

Collecting together, we obtain

$$\begin{aligned} I_1 &= \int_{-T}^{T}\Bigg(-\log\pi-(2B+1)\log\frac{\pi}{2\ell}+2\left(\left(\frac{1}{2}+B\right)\log|t|-\frac{\pi}{2}|t|+\frac{1}{2}\log 2\pi\right) \\ &\quad +\log\left|\mathcal{R}\left(1+B-it;\frac{-k^*}{4\ell}\right)\right|+\pi|t|-\log 2+\mathrm{O}\left(|t|^{-1}\right)\Bigg)\mathrm{d}t \\ &= 2T(2B+1)\log\frac{2\ell}{\pi}+(2B+1)2T\log\frac{T}{e} \\ &\quad +\int_{-T}^{T}\log\left|\mathcal{R}\left(1+B-it;\frac{-k^*}{4\ell}\right)\right|\mathrm{d}t+\mathrm{O}(\log T) \\ &= (2B+1)2T\log\frac{2T\ell}{\pi e}+\int_{-T}^{T}\log\left|\mathcal{R}\left(1+B-it;\frac{-k^*}{4\ell}\right)\right|\mathrm{d}t+\mathrm{O}(\log T). \end{aligned}$$

The integral above looks similar to I_2. We estimate them as we do now for I_2. Note that

$$\frac{1}{4}\exp\left(-2\pi i\frac{k}{4\ell}\right)\mathcal{R}\left(s;\frac{k}{4\ell}\right)=1+\frac{1}{4}\sum_{n=2}^{\infty}\frac{r(n)}{n^s}\exp\left(2\pi i\frac{(n-1)k}{4\ell}\right).$$

This yields

$$-I_2=\int_{-T}^{T}\log\left|1+\frac{1}{4}\sum_{n=2}^{\infty}\frac{r(n)}{n^{A+it}}\exp\left(\frac{\pi ik}{2\ell}(n-1)\right)\right|\mathrm{d}t-2T\log 4.$$

The absolute value of the sum appearing in the latter formula is less than 1. By the Taylor expansion of the logarithm we may bound the integral by

$$\begin{aligned} &\int_{-T}^{T}\mathrm{Re}\left(\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\left(\frac{1}{4}\sum_{n=2}^{\infty}\frac{r(n)}{n^{A+it}}\exp\left(\frac{\pi ik}{2\ell}(n-1)\right)\right)^j\right)\mathrm{d}t \\ &= \mathrm{Re}\sum_{j=1}^{\infty}\frac{(-1)^{j-1}}{j}\frac{1}{4}^j\sum_{n_1=2}^{\infty}\cdots\sum_{n_j=2}^{\infty}\frac{r(n_1)\cdots r(n_j)}{(n_1\cdots n_j)^A}\times \\ &\qquad \times\exp\left(\frac{\pi ik}{2\ell}(n_1+\ldots+n_j-j)\right)\int_{-T}^{T}\frac{\mathrm{d}t}{(n_1\cdots n_j)^{it}} \\ &\ll \sum_{j=1}^{\infty}\frac{1}{j}\sum_{n=2}^{\infty}\left(\frac{r(n)}{4n^A}\right)^j, \end{aligned}$$

and this is bounded. In a similar way we find

$$\int_{-T}^{T}\log\left|\mathcal{R}\left(1+B-it;\frac{-k^*}{4\ell}\right)\right|\mathrm{d}t=-2T\log 4+\mathrm{O}(1).$$

Thus we get

$$I_1 + I_2 = 2T(2B+1)\log\frac{2T\ell}{\pi e} + \mathrm{O}(\log T).$$

It remains to estimate the horizontal integrals I_3, I_4. Suppose that $\operatorname{Re}\mathcal{R}\left(\sigma+iT;\frac{k}{4\ell}\right)$ has N zeros for $-B \leqslant \sigma \leqslant A$. Then divide $[-B, A]$ into at most $N+1$ subintervals in each of which $\operatorname{Re}\mathcal{R}\left(\sigma+iT;\frac{k}{4\ell}\right)$ is of constant sign. Then

$$\left|\arg\mathcal{R}\left(\sigma+iT;\frac{k}{4\ell}\right)\right| \leqslant (N+1)\pi. \tag{7}$$

To estimate N let

$$g(z) = \frac{1}{2}\left(\mathcal{R}\left(z+iT;\frac{k}{4\ell}\right) + \overline{\mathcal{R}\left(\overline{z}+iT;\frac{-k}{4\ell}\right)}\right).$$

Then we have $g(\sigma) = \operatorname{Re}\mathcal{R}\left(\sigma+iT;\frac{k}{4\ell}\right)$. Let $R = A+B$ and choose T so large that $T > 2R$. Now, $\operatorname{Im}(z+iT) > 0$ for $|z-A| < T$. Thus $\mathcal{R}\left(z+iT;\frac{k}{4\ell}\right)$, and hence $g(z)$, is analytic for $|z-A| < T$. Let $n(r)$ denote the number of zeros of $g(z)$ in $|z-A| \leqslant r$. Obviously, we have

$$\int_0^{2R}\frac{n(r)}{r}\mathrm{d}r \geqslant n(R)\int_R^{2R}\frac{\mathrm{d}r}{r} = n(R)\log 2.$$

With Jensen's formula (see [5], §3.61),

$$\int_0^{2R}\frac{n(r)}{r}\mathrm{d}r = \frac{1}{2\pi}\int_0^{2\pi}\log\left|g\left(A+2Re^{i\theta}\right)\right|\mathrm{d}\theta - \log|g(A)|,$$

we deduce

$$n(R) \leqslant \frac{1}{2\pi\log 2}\int_0^{2\pi}\log\left|g\left(A+2Re^{i\theta}\right)\right|\mathrm{d}\theta - \frac{\log|g(a)|}{\log 2}.$$

By (4) it follows that $\log|g(A)|$ is bounded. To bound the integrand above, note that we have by Stirling's formula, the functional equation (3) and (4)

$$\mathcal{R}\left(s;\frac{k}{4\ell}\right) \ll |t|^{1-2\sigma}$$

for $\sigma < 0$, where the implicit constant depends only on l. Using the Phragmèn-Lindelöf principle (see [5], §5.65), we get in any vertical strip of bounded width

$$\mathcal{R}\left(s;\frac{k}{4\ell}\right) \ll |t|^c$$

with a certain positive constant c. Obviously, the same estimate holds for $g(z)$. Thus, the integral above is $\ll \log T$, and $n(R) \ll \log T$. Since the interval $(-B, A)$ is contained in the disc $|z - A| \leqslant R$, we have $N \leqslant n(R)$. Therefore, with (7), we get

$$|I_4| \leqslant \int_{-B}^{A} \left|\arg \mathcal{R}\left(\sigma + iT; \frac{k}{4\ell}\right)\right| \mathrm{d}\sigma \ll \log T.$$

Obviously, I_3 can be bounded in the same way. Thus Theorem 1 is proved.

References

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Apie naujos dzeta funkcijos nulius

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Straipsnyje nagrinėjama nauja dzeta funkcija $R(s; \frac{k}{4\ell}) = \sum_{n=1}^{\infty} \frac{r(n)}{n^s} \exp\left(2\pi i n \frac{k}{4\ell}\right)$, kur k ir ℓ tokie teigiami sveikieji skaičiai, kad k ir 4ℓ yra tarpusavyje pirminiai, $r(n)$ žymi skaičių būdų, kuriais teigiamą sveiką skaičių n galima išreikšti dviejų sveikųjų skaičių kvadratų suma, ir įrodoma šios funkcijos netrivialiųjų nulių skaičiaus asimptotinė formulė.