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On weak convergence of an approximation of a fractional Brownian motion

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In this paper we consider a problem of approximation of a fractional Brownian motion (fBm). Several schemes of such approximations are considered in [1,2,3,5,6]. In this note we extend Sottinen's result [5].

A fBm with the Hurst index $0 < H < 1$ is a centered Gaussian process $X = \{X_t, t \geqslant 0\}$ with $X_0 = 0$ and with the covariance

$$\mathrm{Cov}\big(X_t, X_s\big) = \frac{1}{2}\,\mathrm{Var}(X_1)\big(t^{2H} + s^{2H} - |t-s|^{2H}\big),$$

for all $t, s \geqslant 0$. If $\mathrm{Var}(X_1) = 1$, we write $X = B^H$. If $H > 1/2$, then we have the following kernel representation of B^H with respect to the standard Brownian motion

$$B_t^H = \int_0^t K_H(t,s)\,\mathrm{d}W_s$$

with a deterministic kernel

$$K_H(t,s) = c_H s^{1/2-H} \int_s^t u^{H-1/2}(u-s)^{H-3/2}\,\mathrm{d}u,$$

where c_H is the normalizing constant

$$c_H = \sqrt{\frac{H(2H-1)}{\beta(2-2H, H-1/2)}}\,,$$

$\beta(\cdot,\cdot)$ is the beta function.

Let $\varkappa^n = \{t_k^n\colon\ 0 \leqslant k \leqslant n\}$ be a sequence of partitions of the interval $[0,1]$ such that $t_k^n = k/n$. Denote $\rho^n(t) = [nt]/n$. Define

$$A_t^n = \sum_{k=1}^{[nt]} K_H(\rho^n(t), t_k^n)\,\frac{\xi_k^n}{\sqrt{n}}\,, \qquad M_t^n = \frac{1}{\sqrt{n}}\,\sum_{k=1}^{[nt]} \xi_k^n,$$

where $[nt]$ denotes the integer part of nt, $\{\xi_k^n\}$ is an i.i.d. random variables with $\mathbf{E}\xi_1^n = 0$ and $\mathbf{D}\xi_1^n = 1$.

From the equality

$$K_H(t,s) = c_H(t-s)^\alpha \int_0^1 \Big[\Big(\frac{t}{s}-1\Big)v+1\Big]^\alpha v^{\alpha-1}\mathrm{d}v, \qquad s<t,$$

we see that $K_H(t,s)$ is continuous in $\{0<s<t\}$, $K_H(\cdot,s)$ is increasing and $K_H(t,\cdot)$ is decreasing, where $\alpha = H-1/2$.

Lemma 1. (cf. [5]) *The finite-dimensional distributions of* (M^n, A^n) *converges to those of* (W, B^H).

Proof. Let $\lambda_k, \mu_k \in \mathbb{R}$ for $1 \leqslant k \leqslant m$ and $t_1,\dots,t_m \in (0,1]$ be arbitrary. We want to show that

$$X^n := \sum_{i=1}^m \big(\lambda_i M^n(t_i) + \mu_i A^n(t_i)\big)$$

converges to a normal distribution with variance

$$\mathbf{E}\bigg\{\sum_{i=1}^m \big(\lambda_i W(t_i) + \mu_i B^H(t_i)\big)\bigg\}^2.$$

Note that

$$\begin{aligned}
\mathbf{D}X^n &= \mathbf{E}\bigg\{\sum_{i=1}^m\sum_{k=1}^{[nt_i]}\Big(\lambda_i + \mu_i K_H\big(\rho^n(t_i),t_k^n\big)\Big)\frac{\xi_k^n}{\sqrt{n}}\bigg\}^2\\
&= \frac{1}{n}\sum_{i,j=1}^m \sum_{k=1}^{[n(t_i\wedge t_j)]}\Big(\lambda_i + \mu_i K_H\big(\rho^n(t_i),t_k^n\big)\Big)\Big(\lambda_j + \mu_j K_H\big(\rho^n(t_j),t_k^n\big)\Big)\\
&= \sum_{i,j=1}^m \lambda_i\lambda_j\frac{[n(t_i\wedge t_j)]}{n} + \frac{1}{n}\sum_{i,j=1}^m \mu_i\lambda_j \sum_{k=1}^{[n(t_i\wedge t_j)]} K_H\big(\rho^n(t_i),t_k^n\big)\\
&\quad + \frac{1}{n}\sum_{i,j=1}^m \mu_j\lambda_i \sum_{k=1}^{[n(t_i\wedge t_j)]} K_H\big(\rho^n(t_j),t_k^n\big)\\
&\quad + \frac{1}{n}\sum_{i,j=1}^m \mu_i\mu_j \sum_{k=1}^{[n(t_i\wedge t_j)]} K_H\big(\rho^n(t_i),t_k^n\big)K_H\big(\rho^n(t_j),t_k^n\big). \qquad (1)
\end{aligned}$$

We have

$$\begin{aligned}
\big|K_H(t,s) - K_H\big(\rho^n(t),s\big)\big| &= c_H s^{-\alpha}\int_{\rho^n(t)}^t u^\alpha(u-s)^{\alpha-1}\,\mathrm{d}u\\
&\leqslant \frac{c_H}{\alpha}\Big(\frac{t}{s}\Big)^\alpha\big(t-\rho^n(t)\big)^\alpha \leqslant \Big(\frac{t}{s}\Big)^\alpha\frac{c_H}{\alpha\, n^\alpha}.
\end{aligned}$$

Let $s \in (t_{i-1}^n, t_i^n]$. Then

$$\begin{aligned}
&\big|K_H\big(\rho^n(t), t_i^n\big) - K_H\big(\rho^n(t), s\big)\big| \\
&\quad \leqslant c_H\big|\big(\rho^n(t) - t_i^n\big)^\alpha - \big(\rho^n(t) - s\big)^\alpha\big| \int_0^1 \Big[\Big(\frac{\rho^n(t)}{t_i^n} - 1\Big)v + 1\Big]^\alpha v^{\alpha-1}\mathrm{d}v \\
&\qquad + c_H\big(\rho^n(t) - s\big)^\alpha \int_0^1 \bigg|\Big[\Big(\frac{\rho^n(t)}{t_i^n} - 1\Big)v + 1\Big]^\alpha \\
&\qquad\qquad - \Big[\Big(\frac{\rho^n(t)}{s} - 1\Big)v + 1\Big]^\alpha\bigg| v^{\alpha-1}\mathrm{d}v := I_1 + I_2.
\end{aligned}$$

Further

$$I_1 \leqslant c_H(t_i^n - s)^\alpha \int_0^1 \Big[\Big(\frac{t}{t_i^n} - 1\Big)v + 1\Big]^\alpha v^{\alpha-1}\mathrm{d}v \leqslant \frac{c_H}{\alpha\, n^\alpha}\Big[\Big(\frac{1}{t_i^n}\Big)^\alpha + 1\Big],$$

$$I_2 \leqslant c_H \int_0^1 \Big|\frac{\rho^n(t)}{t_i^n} - \frac{\rho^n(t)}{s}\Big|^\alpha v^{2\alpha-1}\mathrm{d}v \leqslant \frac{c_H}{2\alpha}\Big(\frac{\rho^n(t)(t_i^n - s)}{s t_i^n}\Big)^\alpha \leqslant \frac{c_H}{\alpha\, n^\alpha}\Big(\frac{t}{s}\Big)^\alpha.$$

Assume that $s > \rho^n(\varepsilon)$. Then

$$\bigg|K_H(t,s) - \sum_{i=[n\varepsilon]+1}^{[nt]} K_H\big(\rho^n(t), t_i^n\big)\mathbf{1}_{(t_{i-1}^n, t_i^n]}(s)\bigg| \leqslant \frac{c_H}{\alpha\, n^\alpha}\Big\{3\Big(\frac{1}{\rho^n(\varepsilon)}\Big)^\alpha + 1\Big\}. \quad (2)$$

Note that

$$|K_H(t,s)| \leqslant c_H\alpha^{-1}s^{-\alpha} := \widehat{c}_H\, s^{-\alpha}. \quad (3)$$

Let $\varepsilon \leqslant t_1$. From (2) and (3) we get

$$\begin{aligned}
&\bigg|\frac{1}{n}\sum_{k=1}^{[n(t_i\wedge t_j)]} K_H\big(\rho^n(t_i), t_k^n\big) - \int_0^{t_i\wedge t_j} K_H(t_i, s)\,\mathrm{d}s\bigg| \\
&\quad \leqslant \bigg|\frac{1}{n}\sum_{k=1}^{[n\varepsilon]} K_H\big(\rho^n(t_i), t_k^n\big)\bigg| + \bigg|\int_0^{\rho^n(\varepsilon)} K_H(t_i, s)\,\mathrm{d}s\bigg| \\
&\qquad + \sum_{k=[n\varepsilon]+1}^{[n(t_i\wedge t_j)]}\bigg|\frac{1}{n}K_H\big(\rho^n(t_i), t_k^n\big) - \int_{t_{k-1}^n}^{t_k^n} K_H(t_i, s)\,\mathrm{d}s\bigg| \\
&\qquad + \bigg|\int_{[n(t_i\wedge t_j)]/n}^{t_i\wedge t_j} K_H(t_i, s)\,\mathrm{d}s\bigg| \\
&\quad \leqslant 2\widehat{c}_H\,\frac{\varepsilon^{1-\alpha}}{1-\alpha} + \frac{c_H}{\alpha\, n^\alpha}\Big\{3\Big(\frac{1}{\rho^n(\varepsilon)}\Big)^\alpha + 1\Big\} + \widehat{c}_H(1-\alpha)^{-1}\frac{1}{n^{1-\alpha}}.
\end{aligned}$$

Thus the term on the left-hand side of the above inequality vanishes as $n \to \infty$.

Similarly one can prove that the second and fourth terms in (1) converges to the corresponding integrals. Thus $\sigma_n^2 := \mathbf{D}X^n \to \mathbf{D}\sum_{i=1}^m (\lambda_i W(t_i) + \mu_i B^H(t_i))$ as $n \to \infty$.

Let us write X^n as a sum in k. Denote

$$c_{k,i} = \begin{cases} 1, & k \leqslant [nt_i], \\ 0, & k > [nt_i]. \end{cases}$$

Note that the function $K_H(t,s) = 0$ if $s > t$. Then

$$X^n = \sum_{k=1}^{[nt_m]} \left\{ \sum_{i=1}^m \Big(c_{k,i}\lambda_i + \mu_i K_H\big(\rho^n(t_i), t_k^n\big)\Big)\right\} \frac{\xi_k^n}{\sqrt{n}} = \sum_{k=1}^{[nt_m]} X_k^n.$$

Now we verify the Lindeberg's condition

$$\lim_{n\to\infty} \frac{1}{\sigma_n^2} \sum_{k=1}^{[nt_m]} \mathbf{E}(X_k^n)^2 \mathbf{1}_{\{|X_k^n| > \varepsilon\sigma_n\}} = 0,$$

where $\varepsilon > 0$. Using the properties of the function $K_H(\cdot,\cdot)$ we get

$$\begin{aligned}
(X_k^n)^2 &\leqslant \frac{(\xi_k^n)^2}{n} \left\{ \sum_{i=1}^m \lambda_i + K_H\big(\rho^n(t_m), t_k^n\big) \sum_{i=1}^m \mu_i \right\}^2 \\
&\leqslant 2\,\frac{(\xi_k^n)^2}{n} \left\{ \left(\sum_{i=1}^m \lambda_i\right)^2 + \left(\sum_{i=1}^m \mu_i\right)^2 K_H^2\big(\rho^n(t_m), t_k^n\big) \right\} \\
&\leqslant 2A\,\frac{(\xi_k^n)^2}{n} \big(K_H^2\big(\rho^n(t_m), t_1^n\big) + 1\big) = 2A(\xi_k^n)^2 \delta_n,
\end{aligned} \tag{4}$$

where

$$A := \max\left\{ \left(\sum_{i=1}^m \lambda_i\right)^2, \left(\sum_{i=1}^m \mu_i\right)^2 \right\}, \qquad \delta_n := \frac{1}{n}\big(K_H^2\big(\rho^n(t_m), t_1^n\big) + 1\big).$$

We obtain

$$\big\{|X_k^n| > \varepsilon\sigma_n\big\} \subseteq \big\{2A(\xi_k^n)^2\delta_n > \varepsilon^2\sigma_n^2\big\} =: A^n(\xi_k^n). \tag{5}$$

Using inequality (4) and the inclusion (5) we obtain

$$\mathbf{E}(X_k^n)^2 \mathbf{1}_{\{|X_k^n| > \varepsilon\sigma_n\}} \leqslant \sigma_{n,k}^2 \mathbf{E}(\xi_k^n)^2 \mathbf{1}_{A^n(\xi_k^n)} = \sigma_{n,i}^2 \mathbf{E}\xi^2 \mathbf{1}_{A^n},$$

where $\xi := \xi_1^1$, $A^n := A^n(\xi_1^1)$, and $\sigma_{n,k}^2 = \mathbf{D}X_k^n$. Thus

$$\frac{1}{\sigma_n^2} \sum_{k=1}^{[nt_m]} \mathbf{E}(X_k^n)^2 \mathbf{1}_{\{|X_k^n| > \varepsilon\sigma_n\}} \leqslant \frac{\sigma_{n,1}^2 + \cdots + \sigma_{n,[nt_m]}^2}{\sigma_n^2} \mathbf{E}\xi^2 \mathbf{1}_{A^n} = \mathbf{E}\xi^2 \mathbf{1}_{A^n}.$$

Since $\delta_n \to \infty$ as $n \to \infty$, then $\mathbf{E}\xi^2 \mathbf{1}_{A^n}$ tends to zero. Thus we obtain the convergence of corresponding finite-dimensional distributions. □

For weak convergence of (M^n, A^n) to (W, B^H) it remains to prove the tightness of (M^n, A^n).

First we prove the following lemma.

Lemma 2. *Assume that* $\{\xi_k^n\}$ *are standard normal random variables. For every* $n \geqslant 1$, $\beta \geqslant 1$, *and* $0 < s < t$ *there is a constant* $C = C(H, \beta)$ *such that*

$$\mathbf{E}\big|A^n(t) - A^n(s)\big|^\beta \leqslant C\Big(\frac{[nt]}{n} - \frac{[ns]}{n}\Big)^{\beta(H-1/2)}. \tag{6}$$

Proof. Note that $A^n(t) - A^n(s)$ is a centered Gaussian random variable for every $n \geqslant 1$. Thus it is enough to prove that for some constant C_1

$$\mathbf{E}\big|A^n(t) - A^n(s)\big|^2 \leqslant C_1\Big(\frac{[nt]}{n} - \frac{[ns]}{n}\Big)^{2H-1}. \tag{7}$$

By simple calculations we get

$$\begin{aligned}
\mathbf{E}\big|A^n(t) - A^n(s)\big|^2 &= \frac{1}{n}\sum_{k=[ns]+1}^{[nt]} K_H^2\big(\rho^n(t), t_k^n\big) \\
&\quad + \frac{1}{n}\sum_{k=1}^{[ns]}\Big\{K_H\big(\rho^n(t), t_k^n\big) - K_H\big(\rho^n(s), t_k^n\big)\Big\}^2 \\
&\leqslant c_H^2\,\frac{t^{2H-1}}{n}\sum_{k=[ns]+1}^{[nt]}(t_k^n)^{-2\alpha}\Big\{\int_{t_k^n}^{\rho^n(t)}(u - t_k^n)^{\alpha-1}\,\mathrm{d}u\Big\}^2 \\
&\quad + c_H^2\,\frac{t^{2H-1}}{n}\sum_{k=1}^{[ns]}(t_k^n)^{-2\alpha}\Big\{\int_{\rho^n(s)}^{\rho^n(t)}(u - t_k^n)^{\alpha-1}\,\mathrm{d}u\Big\}^2 \\
&\leqslant \frac{2c_H^2}{\alpha^2}\,\frac{1}{1-2\alpha}\,\big(\rho^n(t) - \rho^n(s)\big)^{2\alpha}.
\end{aligned}$$

Lemma 3. *Under conditions of Lemma 2 the sequence* $\{(M^n, A^n)\}$ *is tight.*

Proof. We shall use the tightness criterium formulated in Theorem 6.4.1 [4]. From

$$\mathbf{E}|M_\delta^n - M_0^n|^2 + \mathbf{E}|A_\delta^n - A_0^n|^2 \leqslant \frac{[n\delta]}{n} + C\Big(\frac{[n\delta]}{n}\Big)^{2\alpha}$$

we get

$$\lim_{\delta\downarrow 0}\limsup_n \mathbf{P}(|(M_\delta^n, A_\delta^n) - (M_0^n, A_0^n)| > \varepsilon) = 0$$

for all $\varepsilon > 0$. It is evident that for every $\lambda > 0$, $s < r < t$

$$\begin{aligned}&\mathbf{P}\big(|(M_r^n,A_r^n)-(M_s^n,A_s^n)|\geqslant\lambda,|(M_t^n,A_t^n)-(M_r^n,A_r^n)|\geqslant\lambda\big)\\&\quad\leqslant\mathbf{P}\big(|M_r^n-M_s^n||M_t^n-M_r^n|+|M_r^n-M_s^n||A_t^n-A_r^n|\\&\qquad+|A_r^n-A_s^n||M_t^n-M_r^n|+|A_r^n-A_s^n||A_t^n-A_r^n|\geqslant\lambda^2\big).\end{aligned}$$

Let $\beta>(2\alpha)^{-1}$. By inequality (6) and the fact that the increments of the process M^n are independent and Gaussian we get

$$\begin{aligned}I\ &:=\mathbf{E}|M_r^n-M_s^n|^2\mathbf{E}|M_t^n-M_r^n|^2+c\mathbf{E}^{\beta/2}|M_r^n-M_s^n|^2\mathbf{E}^{1/2}|A_t^n-A_r^n|^{2\beta}\\&\quad+c\mathbf{E}^{1/2}|A_r^n-A_s^n|^{2\beta}\mathbf{E}^{\beta/2}|M_t^n-M_r^n|^2+\mathbf{E}^{1/2}|A_r^n-A_s^n|^{2\beta}\mathbf{E}^{1/2}|A_t^n-A_r^n|^{2\beta}\\&\leqslant\ \widetilde{C}\left(\frac{[nt]}{n}-\frac{[ns]}{n}\right)^{2\alpha\beta},\end{aligned}$$

where $c=c(\beta)$, $\widehat{C}$, and $\widetilde{C}$ are constants. If now $t-s\geqslant 1/n$ we have

$$I\leqslant\widetilde{C}\,|2(t-s)|^{2\alpha\beta}. \tag{8}$$

If on the other hand $t-s<1/n$ then either s and r or r and t lie in the same subinterval $[i/n,(i+1)/n)$ for some i. Thus $I=0$. Therefore (8) holds for all $s<r<t$. Thus for some constant C we obtain

$$\begin{aligned}\mathbf{P}\big(|(M_r^n,A_r^n)-(M_s^n,A_s^n)|\ &\geqslant\ \lambda,|(M_t^n,A_t^n)-(M_r^n,A_r^n)|\geqslant\lambda\big)\\&\leqslant\ C(\lambda^{-4}\vee\lambda^{-2\beta})(t-s)^{2\alpha\beta}.\end{aligned}$$

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Trupmeninio Brauno judesio aproksimacijos silpno konvergavimo klausimu

K. Kubilius

Naudojantis trupmeninio Brauno judesio išraiška $B_t^H=\int_0^t K_H(t,s)\,\mathrm{d}W_s$, čia W – standartinis Brauno judesys, $K_H(t,s)$ – neatsitiktinis branduolys, sukonstruota pakankamai paprasta aproksimacija, kuri silpnai konverguoja į trupmeninį Brauno judesį.