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Asymptotic distribution of a renewal function estimate

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1. The problem

Suppose that a company has a number of vehicles which have m wheels in all. The problem is to predict, at the given moment t_0, the minimal number n^* of spare tires needed for uninterrupted functioning of vehicles in a given time interval $(t_0, t_0+t]$.

To formulate the problem more precisely, denote by $N_i(t)$ the number of tire failures on the ith wheel in the time interval $(t_0, t_0+t]$ and set $N(t) = \sum_{i=1}^{m} N_i(t)$. We have to find the minimal number n^* such that

$$\mathrm{P}\left(N(t) \leqslant n^*\right) \geqslant \gamma, \tag{1.1}$$

where γ is near 1.

Suppose that processes $N_i(\cdot)$, $i \geqslant 1$, are independent and identically distributed. Then, for m sufficiently large, the probability in the left-hand side of (1.1) is approximated by

$$\Phi\left(\frac{n^* - m\mu(t)}{\sqrt{m}\sigma(t)}\right), \tag{1.2}$$

where Φ denotes the cumulative distribution function (c.d.f.) of the standard normal law, $\mu(t) = \mathrm{E}N_i(t)$ and $\sigma^2(t) = \mathrm{Var}N_i(t)$.

Next, suppose that the time intervals between successive tire failures on a concrete wheel are independent and identically distributed random variables with c.d.f. F. From the renewal theory then follows that

$$\mu(t) = G(t) + \int_0^t G(t-s)\mathrm{d}H(s), \tag{1.3}$$

and

$$\sigma^2(t) = 2\int_0^t \mu(t-s)\mathrm{d}H(s) + \mu(t) - \mu^2(t), \tag{1.4}$$

where

$$G(t) = F(t_0 + t) - \int_0^{t_0} \big(1 - F(t_0 + t - s)\big) \mathrm{d}H(s). \tag{1.5}$$

In (1.3)–(1.5), H denotes the so-called *renewal function*, which is defined by $H(t) = \sum_{k=1}^{\infty} F_k(t)$, where $F_k = F * \cdots * F$ is the k-fold convolution of the function F.

By (1.1)–(1.5), the problem of estimation of n^* is reduced to estimating the renewal function. In this paper we consider the estimation of $H(t)$ for a fixed t.

Suppose that we have observed n independent random variables $T_1, \ldots, T_n$, which are identically distributed according to the c.d.f. F. Then the distribution functions F_k can be estimated by corresponding empirical distribution functions

$$\widehat{F}_{kn}(t) = n^{-k} \sum_{i_1, \ldots, i_k = 1}^{n} 1_{\{T_{i_1} + \cdots + T_{i_k} \leqslant t\}}. \tag{1.6}$$

Set $\widehat{H}_n(t) = \sum_{k=1}^{\infty} \widehat{F}_{kn}(t)$. The problem is to find asymptotic distribution of the random variable

$$\sqrt{n}\big(\widehat{H}_n(t) - H(t)\big); \tag{1.7}$$

here t is a fixed positive number.

We do not know any theoretical result about estimating the renewal function. So we hope the following theorem is new.

Theorem 1.1. *If the distribution function F is continuous, then* (1.7) *tends in distribution to the random variable*

$$W(t) + \int_0^t \sum_{k=1}^{\infty} (k+1) F_k(t - s) \mathrm{d}W(s), \tag{1.8}$$

where W is a Gausian zero mean random process with the covariance function

$$\mathrm{E}[W(s)W(s')] = F(s \wedge s') - F(s)F(s'). \tag{1.9}$$

2. The proof of Theorem 1.1

The plan of the proof. Denote

$$\xi_{kn} = \sqrt{n}\big(\widehat{F}_{kn}(t) - F_k(t)\big)$$

and $\xi_n = (\xi_{1n}, \xi_{2n}, \xi_{3n}, \ldots)$. The proof is performed in 4 steps.

Step 1. We prove that ξ_n is a random element of ℓ_1, the Banach space of all summable sequences $x = (x_k)$ endowed with the norm

$$\|x\| = \sum_{k=1}^{\infty} |x_k| .$$

Step 2. We find a random element $\zeta = (\zeta_1, \zeta_2, \ldots)$ of ℓ_1 such that, for all $k \geqslant 1$, $(\xi_{1n}, \ldots, \xi_{kn})$ tends to $(\zeta_1, \ldots, \zeta_k)$ in distribution.

Step 3. We prove that the sequence (ξ_n) is tight and therefore it tends in distribution in the space ℓ_1 to ζ.

Step 4. We get the assertion of the theorem by noticing that $x \mapsto \sum_{k=1}^{\infty} x_k$ is a continuous linear functional on ℓ_1.

Step 1. If $\widetilde{F}_1$ and $\widetilde{F}_2$ are c.d.f. of two positive random variables, then

$$(\widetilde{F}_1 * \widetilde{F}_2)(t) = \int_0^t \widetilde{F}_2(t - s_1) \mathrm{d}\widetilde{F}_1(s_1) \leqslant \widetilde{F}_2(t) \int_0^t \mathrm{d}\widetilde{F}_1(s_1) = \widetilde{F}_1(t)\widetilde{F}_2(t).$$

Obviously, for all $k, l \geqslant 1$, $F_{k+l} = F_k * F_l$. Therefore

$$F_{k+l}(t) \leqslant F_k(t) F_l(t). \tag{2.1}$$

In particular, the sequence $\big(F_k(t) \mid k \geqslant 1\big)$ is non-increasing.

Consider the sequence of random variables (S_k) such that S_1, $S_2 - S_1, \ldots$ are independent copies of T. Find a $\delta > 0$ such that $F(\delta) < 1/2$. Fix an arbitrary integer $k_0 \geqslant t/\delta$. Then

$$\mathrm{P}\{S_1 > \delta, S_2 - S_1 > \delta, \ldots, S_{k_0} - S_{k_0 - 1} > \delta\} \leqslant \mathrm{P}\{S_{k_0} > t\} = 1 - F_{k_0}(t),$$

and therefore

$$F_{k_0}(t) \leqslant 1 - \big[1 - F(\delta)\big]^{k_0} < 1 - 2^{-k_0}.$$

Together with (2.1), this yields the estimate

$$F_k(t) \leqslant q^{[k/k_0]} \tag{2.2}$$

($k = 1, 2, \ldots$), where $q = 1 - 2^{-k_0} < 1$ and $[x]$ stands for the integer part of x. Therefore $(F_k(t) \mid k \geqslant 1) \in \ell_1$.

It is easily seen from (1.6) that almost surely $\widehat{F}_{kn}(t) = 0$ for k sufficiently large (and for each fixed n). Hence $\xi_n \in \ell_1$ almost surely.

Step 2. Suppose that F is continuous and define $\widehat{F}_n = \widehat{F}_{1n}$, where $\widehat{F}_{1n}$ is given by (1.6). It is well-known that $\sqrt{n}(\widehat{F}_n - F)$ tends in distribution in the Skorokhod space $D[0; \infty)$ to the continuous zero mean Gaussian process W with the covariance function (1.9). Therefore by Theorem 5.5 of [1] we get the following result.

PROPOSITION. Let φ be a Borel function defined on some Borel subset $D \subset D[0;\infty)$ with values in $\mathbb{R}^k$. Suppose that

$$\sqrt{n}\big(\varphi(F^{(n)}) - \varphi(F)\big) \to \varphi'(\Delta F)$$

for each sequence $(F^{(n)}) \subset D$ such that $\Delta F^{(n)}$ converges to ΔF uniformly on bounded subsets of $[0;\infty)$; here $\Delta F^{(n)} = \sqrt{n}(F^{(n)} - F)$, $\Delta F \in C[0;\infty)$ and φ' is some Borel function on $C[0;\infty)$. Then

$$\sqrt{n}\big(\varphi(\widehat{F}^{(n)}) - \varphi(F)\big) \xrightarrow{d} \varphi'(W).$$

We apply this result to the function $\varphi = (\varphi_1, \dots, \varphi_k)$ defined as follows. Let D be the set of all distribution functions corresponding to positive random variables and, for $\widetilde{F} \in D$, $\varphi_j(\widetilde{F}) = (\underbrace{\widetilde{F} * \cdots * \widetilde{F}}_{j \text{ times}})(t)$, $j = 1, \dots, k$. Suppose $(F^{(n)}) \subset D$, $\Delta F^{(n)} = \sqrt{n}(F^{(n)} - F)$ and $\Delta F^{(n)}$ converges to some continuous function ΔF uniformly on compact subsets of $[0;\infty)$. For $j \geqslant 1$ denote $F_j^{(n)} = \underbrace{F^{(n)} * \cdots * F^{(n)}}_{j \text{ times}}$ and set $F_0^{(n)} = F_0 = I$, where $I(t) = 1_{[0;\infty)}(t)$ (this is the distribution function of the degenerate at point 0 random variable). Obviously,

$$\begin{aligned}\sqrt{n}\big(\varphi_j(F^{(n)}) - \varphi(F)\big) &= \sqrt{n}\sum_{i=1}^{j}\big((F_i^{(n)} * F_{j-i})(t) - (F_{i-1}^{(n)} * F_{j-i+1})(t)\big)\\ &= \sum_{i=1}^{j}\int_0^t f_{in}(s)\,\mathrm{d}F_{i-1}^{(n)}(s),\end{aligned}$$

where, for $s \in [0;t]$,

$$f_{in}(s) = \int_0^{t-s} \Delta F^{(n)}(t - s - s')\mathrm{d}F_{j-i}(s').$$

Moreover, $f_{in}(s) \to f_i(s)$ uniformly in $[0;t]$, where

$$f_i(s) = \int_0^{t-s} \Delta F(t - s - s')\mathrm{d}F_{j-i}(s').$$

Notice that, since ΔF is bounded on $[0;t]$, f_i is continuous at each point $s \in [0;t]$.

Let $T^{(n)}$ denote the random variable with the distribution function $F^{(n)}$ and let $T_1^{(n)}, T_2^{(n)}, \dots$ be its independent copies. Since $F^{(n)}(t) \to F(t)$ for all t, $T^{(n)} \xrightarrow{d} T$ and therefore $T_1^{(n)} + \cdots + T_{i-1}^{(n)} \xrightarrow{d} S_{i-1}$. By continuity of f_i,

$$\int_0^t f_i(s)\mathrm{d}F_{i-1}^{(n)}(s) \to \int_0^t f_i(s)\mathrm{d}F_{i-1}(s).$$

On the other hand,

$$\left| \int_0^t f_{in}(s) \mathrm{d}F_{i-1}^{(n)}(s) - \int_0^t f_i(s) \mathrm{d}F_{i-1}^{(n)}(s) \right| \leqslant \sup_{s \in [0;t]} |f_{in}(s) - f_i(s)| \to 0.$$

Hence

$$\int_0^t f_{in}(s) \mathrm{d}F_{i-1}^{(n)}(s) \to \int_0^t f_i(s) \mathrm{d}F_{i-1}(s),$$

and therefore

$$\sqrt{n}\big(\varphi_j(F^{(n)}) - \varphi(F)\big) \to \varphi_j'(\Delta F),$$

where

$$\varphi_j'(\Delta F) = \sum_{i=1}^{j} \int_0^t f_i(s) \mathrm{d}F_{i-1}(s) = j \int_0^t \Delta F(t-s) \mathrm{d}F_{j-1}(s).$$

For $j \geqslant 1$ set $\zeta_j = j \int_0^t W(t-s) \mathrm{d}F_{j-1}(s)$ and let $\zeta = (\zeta_1, \zeta_2, \ldots)$. By Proposition, for all $k \geqslant 1$, $(\xi_{1n}, \ldots, \xi_{kn}) \xrightarrow{d} (\zeta_1, \ldots, \zeta_k)$ and it remains to prove that $\zeta \in \ell_1$ a.s. This follows from (2.2) and the estimate

$$|\zeta_j| \leqslant \sup_{s \in [0;t]} |W(s)| \, j F_{j-1}(t).$$

Step 3. Similarly as above we get

$$\begin{aligned} |\xi_{kn}| &= \left| \sqrt{n} \sum_{j=1}^{k} \int_0^t \mathrm{d}\widehat{F}_{j-1,n}(s) \int_0^{t-s} \big(\widehat{F}_n(t-s-s') - F(t-s-s')\big) \, \mathrm{d}F_{k-j}(s') \right| \\ &\leqslant \Delta_n \sum_{j=1}^{k} \widehat{F}_{j-1,n}(t) F_{k-j}(t), \end{aligned} \tag{2.3}$$

where $\Delta_n = \sqrt{n} \sup_{s \in [0;t]} \left| \widehat{F}_n(s) - F(s) \right|$. It is well-known [1] that Δ_n tends in distribution to $\sup_{s \in [0;t]} |W(s)|$ and therefore it is bounded in probability:

$$\sup_n \mathrm{P}\{\Delta_n > c\} \to 0,$$

as $c \to \infty$.

Fix an $\varepsilon > 0$ and find a $c < \infty$ such that $\mathrm{P}\{\Delta_n > c\} < \varepsilon$ for all n. Let δ, k_0 and q be the same as in Step 1. Find n_0 such that, for all $n \geqslant n_0$,

$$\mathrm{P}\{\widehat{F}_n(\delta) < 1/2\} \geqslant 1 - \varepsilon.$$

Similarly as in Step 1 we get that inequality $\widehat{F}_n(\delta) < 1/2$ implies $\widehat{F}_{kn}(t) \leqslant q^{[k/k_0]}$ for all k. Hence, for all $n \geqslant n_0$,

$$\mathrm{P}\{\forall k \geqslant 1 \ \widehat{F}_{kn}(t) \leqslant q^{[k/k_0]}\} \geqslant 1 - \varepsilon$$

and, by (2.3),

$$\mathrm{P}\{\forall k \geqslant 1 \ |\xi_{kn}| \leqslant ckq^{[(k-2)/(2k_0)]}\} \geqslant 1 - 2\varepsilon.$$

Set

$$K = \{x \in \ell_1 \mid \forall k \geqslant 1 \ |x_k| \leqslant ckq^{[(k-2)/(2k_0)]}\}.$$

Then $\mathrm{P}\{\xi_n \in K\} \geqslant 1 - 2\varepsilon$. On the other hand, K is a compact subset of ℓ_1. By the Prokhorov theorem, the sequence (ξ_n) is tight.

Step 4. It is well-known that $x \mapsto \sum_{k=1}^{\infty} x_k$ is a continuous linear functional on ℓ_1. Therefore the results of Steps 1–3 imply that (1.7) tends in distribution to the random variable

$$\sum_{k=1}^{\infty} k \int_0^t W(t-s)\mathrm{d}F_{k-1}(s).$$

By the definition of the stochastic integral, the limit random variable can be written in the form (1.8).

References

[1] P. Billingsley, *Convergence of Probability Measures*, Wiley, New York (1968).

Atstatymo funkcijos įverčio asimptotinis skirstinys

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Patikimumo teorijos modeliuose, kuriuose sugedusios detalės pakeičiamos naujomis, daugelis patikimumo charakteristikų išreiškiamos per atstatymo funkciją. Šiame straipsnyje pasiūlytas atstatymo funkcijos įvertis ir apskaičiuota to įverčio reikšmės tam tikrame taške asimptotinė dispersija.