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On the identification of Wiener systems

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1. Introduction

A lot of physical systems are naturally described as Wiener systems with piecewise linear nonlinearity, i.e., when the linear system is followed by a hard nonlinearity-like saturation or dead-zone [1]. A special class of such systems is piecewise affine (PWA) systems, consisting of some subsystems, between which occasional switchings happen at different time moments [2]. Assuming the nonlinearity to be piecewise linear, one could let the nonlinear part of the Wiener system be represented by different regression functions with some parameters, that are unknown beforehand. In such a case, observations of the output of a Wiener system could be partitioned into distinct data sets according to different descriptions. The boundaries of sets of observations depend on the value of the unknown threshold a – observations are divided into regimes subject to whether the some observed threshold variable is smaller or larger than a [3]. Thus, there arises a problem, first, to find a way to partition the available data, second, to calculate the estimates of parameters of regression functions by processing particles of observations to be determined, and, third, to get the unknown threshold.

2. Statement of the problem

The Wiener system consists of a linear part $G(q,\mathbf{\Theta})$ followed by a static nonlinearity $f(\cdot,\eta)$. The linear part of the PWA system is dynamic, time invariant, causal, and stable. It can be represented by a time invariant dynamic system (LTI) with the transfer function $G(q,\mathbf{\Theta})$ as a rational function of the form

$$G(q,\mathbf{\Theta}) = \frac{b_1q^{-1}+,\ldots,+b_mq^{-m}}{1+a_1q^{-1}+,\ldots,+a_mq^{-m}} = \frac{B(q,\mathbf{b})}{1+A(q,\mathbf{a})} \tag{1}$$

with a finite number of parameters

$$\mathbf{\Theta}^T{=}(b_1,\ldots,b_m,a_1,\ldots,a_m),\quad \mathbf{b}^T=(b_1,\ldots,b_m),\quad \mathbf{a}^T=(a_1,\ldots,a_m), \tag{2}$$

that are determined from the set $\mathbf{\Omega}$ of permissible parameter values $\mathbf{\Theta}$. Here q is a time-shift operator, the set $\mathbf{\Omega}$ is restricted by conditions on the stability of the respective difference equation. The unknown intermediate signal

$$x(k) = \frac{B(q,\mathbf{b})}{1+A(q,\mathbf{a})}u(k)+v(k), \tag{3}$$

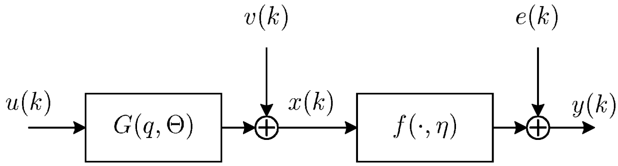

Fig. 1. The PWA system with the process noise $v(k)$ and that of the measurement $e(k)$. The linear dynamic part $G(q, \mathbf{\Theta})$ of the PWA system is parameterised by $\mathbf{\Theta}$, while a static nonlinear part $f(\cdot, \eta)$ – by η. Signals: $u(k)$ is input, $y(k)$ is output, $x(k)$ is an unmeasurable intermediate signal.

generated by a linear part of the PWA system (1) as a response to the input $u(k)$ and corrupted by the additive noise $v(k)$, is acting on a static nonlinear part $f(\cdot, \eta)$ (Fig. 1) with the vector of parameters η , i.e.,

$$y(k) = f(x(k), \eta) + e(k). \tag{4}$$

Here the nonlinear part $f(\cdot, \eta)$ of the PWA system is a saturation-like function of the form [4]

$$f(x(k), \eta) = \begin{cases} c_0 + c_1 x(k) & \text{if } x(k) \leqslant -a, \\ x(k) & \text{if } -a \leqslant x(k) \leqslant a, \\ d_0 + d_1 x(k) & \text{if } x(k) \geqslant a, \end{cases} \tag{5}$$

that could be partitioned into three functions. These functions are: $f\{x(k; \mathbf{\Theta}), \mathbf{c}, a\} = c_0 + c_1 x(k)$, $f\{x(k; \mathbf{\Theta}), a\} = x(k)$, and $f\{x(k; \mathbf{\Theta}), \mathbf{d}, a\} = d_0 + d_1 x(k)$. The function $f\{x(k; \mathbf{\Theta}), \mathbf{c}, a\}$ has only negative values, when $x(k) \leqslant -a$, $f\{x(k; \mathbf{\Theta}), a\}$ has arbitrary positive, as well as negative values, when $-a \leqslant x(k) \leqslant a$, and $f\{x(k; \mathbf{\Theta}), \mathbf{d}, a\}$ has only positive values, when $x(k) \geqslant a$. Here $x(k; \mathbf{\Theta}) \equiv x(k)$, $\mathbf{c}^T = (c_0, c_1)$, $c_0 = -a(1 - c_1)$, $0 < c_1 < a$, $\mathbf{d}^T = (d_0, d_1)$, $d_0 = a(1 - d_1)$, $0 < d_1 < a$.

The process noise $v(k) \equiv \xi(k)$ and the measurement noise $e(k) \equiv \zeta(k)$ are added to an intermediate signal $x(k)$ and the output $y(k)$, respectively, $\xi(k)$, $\zeta(k)$ are noncorrelated between each other sequences of independent Gaussian variables with $E\{\xi(k)\} = 0$, $E\{\zeta(k)\} = 0$, $E\{\xi(k)\xi(k + \tau)\} = \sigma_\xi^2 \delta(\tau)$, $E\{\zeta(k)\zeta(k + \tau)\} = \sigma_\zeta^2 \delta(\tau)$; $E\{\cdot\}$ is a mean value, $\sigma_\zeta^2, \sigma_\xi^2$ are variances of ζ and ξ, respectively, $\delta(\tau)$ is the Kronecker delta function.

The aim of the given paper is to estimate parameters (2) of the linear part (1) of the PWA system, parameters $\eta = (c_0, c_1, d_0, d_1)^T$ of the nonlinear part (5) and the threshold a of nonlinearity (5) by processing N pairs of observations $u(k)$ and $y(k)$.

3. The data rearrangement

At first, let us rearrange the data $y(k)$ in an ascending order of their values. Thus, the observations of the rearranged output $\tilde{y}(k)$ of the PWA system should be partitioned into three data sets: left-hand side data set with values lower or equal to negative a, middle data set with values higher than negative a but lower or equal to a, and right-hand side data set with values higher than a (it is assumed that no less than 50% observations are concentrated on the middle-set and approximately by 25% or less on any side set). The observations with the highest and positive values will be concentrated on the right-hand side set, while the observations with the lowest and negative values on the left-hand side one. The observations of the middle data set (in such a case observations of $\tilde{y}(k)$ coincide with the respective observations of the intermediate signal $x(k)$) will be concentrated now around the time origin. Therefore one could get these observations simply by choosing from the time origin in both directions not less than 25% of observations of $\tilde{y}(k)$.

At second, let us reconstruct an unmeasurable intermediate signal $x(k)$ using the middle data set of $\tilde{y}(k)\ \forall\ k \in \overline{1,\mathrm{N}}$. To calculate the auxiliary signal $\hat{x}(k)$ (the estimate of unmeasurable $x(k)$) one could approximate the model of the linear part of the PWA system (1) by the finite impulse response (FIR) system of the form [5]

$$\tilde{y}(k) = \beta_0 + \beta_1 u(k) + \beta_2 u(k-1) + \ldots + \beta_\nu u(k-\nu+1) + e(k). \tag{6}$$

Here

$$\beta^T = (\beta_0, \beta_1 \ldots, \beta_\nu) \tag{7}$$

is a $(\nu+1)\times 1$ vector of unknown parameters, ν is the order of the FIR filter. In this case, the dependence of some regressors on the process output will be facilitated, and the assumption of the ordinary LS that the regressors depend only on the nonnoisy input signal, will be satisfied. This is the main consequence of replacing the initial transfer function $G(q, \boldsymbol{\Theta})$ of the linear part of the PWA system by the FIR filter (6). Then, the parametric estimation technique, based on ordinary LS, could be applied in the estimation of parameters (7) of given FIR system

$$\widetilde{\mathbf{Y}} = \boldsymbol{\Lambda}\beta, \tag{8}$$

using the rearranged observations of the middle data-set, because the rearrangement of observations does not influence the accuracy of estimates to be calculated (observations of the output $y(k)$ could be rearranged in an ascending order of their values by interchanging equations in the initial system (8)). Here $\widetilde{\mathbf{Y}}$ is the $(\mathrm{L}-\nu)\times 1$ vector of the middle data set of $\tilde{y}(k)$, $\boldsymbol{\Lambda}$ is the $(\mathrm{L}-\nu)\times(\nu+1)$ regression matrix, consisting only of observations of the input $u(k)$, besides, $\mathrm{L} < \mathrm{N}$. To estimate parameters β, one can use the expression

$$\hat{\beta} = \left(\boldsymbol{\Lambda}^T\boldsymbol{\Lambda}\right)^{-1}\boldsymbol{\Lambda}^T\mathbf{Y}, \tag{9}$$

where $\hat{\beta}$ is a $(\nu+1)\times 1$ vector of the estimates of parameters (7). Afterwards, the estimate $\hat{x}(k)$ of the intermediate signal $x(k)$ could be determined using (6), where, instead of the true values (7), their estimates $\hat{\beta}$ are substituted, i.e.,

$$\hat{x}(k) = \hat{\beta}_0 + \hat{\beta}_1 u(k) + \hat{\beta}_2 u(k-1) + \ldots + \hat{\beta}_\nu u(k-\nu+1). \tag{10}$$

The estimates of parameters (2) of the transfer function $G(q,\mathbf{\Theta})$ are calculated according to

$$\hat{\mathbf{\Theta}} = \left(\mathbf{X}^T\mathbf{X}\right)^{-1}\mathbf{X}^T\mathbf{U}, \tag{11}$$

Here

$$\hat{\mathbf{\Theta}}^T = \left(\hat{\mathbf{b}},\hat{\mathbf{a}}\right)^T, \quad \hat{\mathbf{b}}^T = \left(\hat{b}_1,\ldots,\hat{b}_m\right), \quad \hat{\mathbf{a}}^T = (\hat{a}_1,\ldots,\hat{a}_m) \tag{12}$$

are a $2m \times 1, m \times 1, m \times 1$ vectors of the estimates of parameters, respectively, $\mathbf{X}$ is the $(\mathrm{N}_2 - m) \times 2m$ matrix, consisting of observations of the input $u(k)$ and the auxiliary signal $\hat{x}(k)$, and $\mathbf{U}$ is the $(\mathrm{N}_2 - m - 1) \times 1$ vector, consisting of the observations of $\hat{x}(k)$, N_2 is the whole number of observations of the middle data set.

Estimates of the parameters c_0, d_0 and c_1, d_1 are calculated by the ordinary least squares, too. In such a case, the sums of the form

$$I(c_0,c_1) = \sum_{i=1}^{\mathrm{N}_1}\left[\tilde{y}(i) - c_0 - c_1\widetilde{\hat{x}}(i)\right]^2 = \min!, \tag{13}$$

$$I(d_0,d_1) = \sum_{j=\mathrm{N}_2+1}^{\mathrm{N}}\left[\tilde{y}(j) - d_0 - d_1\widetilde{\hat{x}}(j)\right]^2 = \min!, \tag{14}$$

are to be minimized in respect of parameters c_0, c_1 and d_0, d_1, respectively, using side-set data particles of $\tilde{y}(k)$ and observations of the auxiliary signal $\hat{x}(k)$. Here N_1 is the number of observations of the left-side set, respectively, $\widetilde{\hat{x}}(k)$ are the observations of the signal $\hat{x}(k)$ that were rearranged in accordance with $\tilde{y}(k)$.

The estimates of the threshold a for the right-hand side and left-hand side sets are found according to

$$\hat{a} = \hat{d}_0/(1 - \hat{d}_1), \hat{a} = \hat{c}_0/(1 - \hat{c}_1), \tag{15}$$

respectively.

In order to determine how different realizations of process and measurement noises affect the accuracy of estimation of unknown parameters, we have used the Monte Carlo simulation with 10 data samples, each containing 100 input-output observation pairs. 10 experiments with the same realization of the process noise $v(k)$ and different realizations of the measurement noise $e(k)$ of different levels of its intensity were carried out. The intensity of noises was assured by choosing respective signal-to-noise ratios (SNR) (the square root of the ratio of signal and noise variances). For the process noise $\mathrm{SNR}^{\mathrm{v}}$ was equal to 100 and for the measurement noise $\mathrm{SNR}^{\mathrm{e}}$: 1, 10, 100. As inputs for all given nonlinearities, the periodical signal and white Gaussian noise with variance 1 were chosen. In each ith experiment the estimates of parameters were calculated. During the Monte Carlo simulation, averaged values of the estimates of parameters and of the threshold and their confidence intervals were calculated. In Tables 1, 2 for each input the averaged estimates of parameters and threshold a of the simulated PWA system (Fig. 1) with the linear part (1) ($b_1 = 0.3; a_1 = -0.5$) and piecewise nonlinearity (5) ($c_0 = -0.9, c_1 = 0.1, d_0 = 0.9, d_1 = 0.1, a = 1$) with their confidence intervals are presented. It should be noted that in each experiment here the value of SNR^v was fixed and the same while, the values of SNR^e were varying due to different realizations of $e(k)$. The Monte Carlo simulation (Tables 1, 2) implies that the accuracy of parametric identification of the PWA system depends on the intensity of measurement noise.

The problem of identification of PWA systems could be essentially reduced by a simple data rearrangement in an ascending order of their values. Thus, the available data are partitioned into three data sets that correspond to distinct threshold regression models. Later on the estimates of unknown parameters of linear regression models could be calculated by processing the respective sets of input and rearranged output observations.

Table 1

Averaged estimates of the parameters $b_1, a_1, c_0, c_1, d_0, d_1$, and thresholds a, $-a$ with their confidence intervals. Input: the periodical signal. $\mathrm{SNR}^v = 100$

Estimates	$\mathrm{SNR}^e = 1$	$\mathrm{SNR}^e = 10$	$\mathrm{SNR}^e = 100$
$\hat{b}_1$	0.28 ± 0.07	0.3 ± 0.00	0.3 ± 0.00
$\hat{a}_1$	-0.52 ± 0.06	-0.5 ± 0.00	-0.5 ± 0.00
$\hat{c}_0$	-0.85 ± 0.31	-0.89 ± 0.04	-0.9 ± 0.00
$\hat{c}_1$	0.16 ± 0.21	0.1 ± 0.02	0.1 ± 0.00
$\hat{d}_0$	1 ± 0.5	0.89 ± 0.04	0.9 ± 0.00
$\hat{d}_1$	0.04 ± 0.4	0.1 ± 0.02	0.1 ± 0.00
$\hat{a}$	0.97 ± 0.25	1 ± 0.02	1 ± 0.00
$-\hat{a}$	-1 ± 0.22	-1 ± 0.02	-1 ± 0.00

Table 2

The values and notation are the same as in Table 1. Input – the Gaussian white noise

Estimates	$\mathrm{SNR}^e = 1$	$\mathrm{SNR}^e = 10$	$\mathrm{SNR}^e = 100$
$\hat{b}_1$	0.31 ± 0.04	0.3 ± 0.00	0.3 ± 0.00
$\hat{a}_1$	-0.39 ± 0.08	-0.5 ± 0.01	-0.5 ± 0.00
$\hat{c}_0$	-0.5 ± 0.86	-0.84 ± 0.06	-0.89 ± 0.00
$\hat{c}_1$	0.42 ± 0.72	0.1 ± 0.05	0.1 ± 0.00
$\hat{d}_0$	1.07 ± 0.51	0.91 ± 0.06	0.9 ± 0.00
$\hat{d}_1$	0.04 ± 0.44	0.15 ± 0.05	0.1 ± 0.00
$\hat{a}$	1.03 ± 0.4	0.99 ± 0.02	1 ± 0.00
$-\hat{a}$	-1.21 ± 0.19	-1 ± 0.02	-1 ± 0.00

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Apie Vinerio sistemų identifikavimą

R. Pupeikis

Straipsnyje nagrinėjamas Vinerio sistemų laipsniškas tiesinės dalies, aprašomos skirtumine lygtimi su nežinomais koeficientais ir dalimis tiesiško netiesiškumo su nežinomais nuožulnumais bei nežinomu slenksčių junginys. Parodyta, kad pertvarkius išėjimo signalo stebėjimus pagal didėjančias jų reikšmes, galima išskirti vidurinę stebėjimų dalį, atitinkančią nestebimo tarpinio signalo stebėjimus. Pasiūlytas pilno tarpinio signalo atstatymo būdas pagal įėjimo signalo ir išėjimo signalo vidurinės dalies stebėjimus. Nežinomų tiesinės Vinerio sistemos dalies koeficientų ir dalimis tiesiško netiesiškumo parametrų bei slenksčio įverčiai gaunami mažiausiųjų kvadratų metodo algoritmais, apdorojant stebimų įėjimo, pertvarkyto išėjimo bei atkurto tarpinio signalų duomenis. Pateikti modeliavimo rezultatai.