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A discrete universality theorem for the Matsumoto zeta-function

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Let g(m) be a positive integer, $f(j,m)$, $1 \leqslant j \leqslant g(m)$, be positive integers, and let $a_m^{(j)}$ be a complex number. Define a polynomial

$$A_m(x) = \prod_{j=1}^{g(m)} \left(1 - a_m^{(j)} x^{f(j,m)}\right)$$

of degree $f(1,m) + \ldots + f(g(m),m)$. Let $s = \sigma + it$ be a complex variable, and let p_m denote the mth prime number. The Matsumoto zeta-function $\varphi(s)$ is defined by

$$\varphi(s) = \prod_{m=1}^{\infty} A_m^{-1}(p_m^{-s}). \tag{1}$$

This function was introduced by K. Matsumoto in [4]. Later it was studied by K. Matsumoto, A. Laurinčikas and by the author under the conditions

$$g(m) \leqslant c_1 p_m^{\alpha}, \quad |a_m^{(j)}| \leqslant p_m^{\beta} \tag{2}$$

with some non-negative constants c_1, α and β. In this case the infinite product (1) converges absolutely for $\sigma > \alpha + \beta + 1$, and defines there a holomorphic function without zeros.

Note that the Matsumoto zeta-function is a generalization of classical zeta-functions, for example, of the Riemann zeta-function, of zeta-functions attached to cusp form.

In [3] the universality theorem for the Matsumoto zeta-function was proved. The aim of this note is to give a discrete version of this theorem. In the sequel, we suppose that the function $\varphi(s)$ is analytic in the strip $D = \{s \in \mathbb{C}\colon \rho_0 < \sigma < \alpha + \beta + 1\}$ where $\alpha + \beta + \frac{1}{2} < \rho_0 < \alpha + \beta + 1$. Moreover, we assume that for $\sigma \geqslant \rho_0$

$$\varphi(\sigma + it) = O\left(|t|^{c_2}\right), \tag{3}$$

and

$$\int_0^T |\varphi(\rho_0 + it)|^2 \, dt = BT, \quad T \to \infty. \tag{4}$$

Denote

$$M(m) = \sum_{\substack{j=1 \\ f(j,m)=1}}^{g(m)} a_m^{(j)} p_m^{-\alpha-\beta}.$$

Let $h > 0$, and suppose that $\exp\left\{\frac{2\pi k}{h}\right\}$ is an irrational number for all integers $k \neq 0$.

Theorem. *Let the conditions* (2)–(4) *be satisfied. Moreover, we suppose that* $M(m) \geqslant c_3 > 0$ *for all* $m \geqslant 1$. *Let* K *be a compact subset of the strip* D *with connected complement. Let* $f(s)$ *be a non-vanishing continuous function on* K *which is analytic in the interior of* K. *Then for every* $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{ 0 \leqslant m \leqslant N\colon \sup_{s\in K} |\varphi(s+imh) - f(s)| < \varepsilon \right\} > 0.$$

For the proof of the theorem a discrete limit theorem in the space of analytic functions for the function $\varphi(s)$ is used. Let $D_1 = \{s \in \mathbb{C}\colon \sigma > \rho_0\}$, and let $M(D_1)$ denote the space of meromorphic on D_1 functions equipped with the topology of uniform convergence on compacta. Then in [2] the following statement was proved.

Lemma 1. *Let the conditions* (2)–(4) *be satisfied. Then the probability measure*

$$\frac{1}{N+1} \# \{0 \leqslant m \leqslant N\colon \varphi(s+imh) \in A\}, \quad A \in \mathcal{B}(M(D_1)),$$

converges weakly to the measure P_φ *as* $N \to \infty$. *The limit measure* P_φ *is the distribution of the random element*

$$\varphi(s,\omega) = \prod_{m=1}^{\infty} \prod_{j=1}^{g(m)} \left(1 - \frac{\omega^{f(j,m)}(p_m) a_m^{(j)}}{p_m^{s f(j,m)}}\right)^{-1}, \quad s \in D_1, \quad \omega \in \Omega,$$

where

$$\Omega = \prod_{m=1}^{\infty} \gamma_{p_m}, \quad \gamma_{p_m} = \{s \in \mathbb{C}\colon |s| = 1\},$$

and $\omega(p_m)$ *is the projection of* $\omega \in \Omega$ *to the space* γ_{p_m}.

The proof is given in [2].

Now let $H(D)$ be the space of analytic on D functions equipped with a topology of uniform convergence on compacta. Denote by $P_{\varphi,D}$ the restriction of P_φ to the space $(H(D), \mathcal{B}(H(D)))$.

Lemma 2. *Let the conditions* (2)–(4) *be satisfied. Then the probability measure*

$$P_{N,D}(A) = \frac{1}{N+1}\#\{0 \leqslant m \leqslant N\colon \varphi(s+imh) \in A\}, \quad A \in \mathcal{B}(H(D)),$$

converges weakly to $P_{\varphi,D}$ as $N \to \infty$.

Proof. Let the function $F\colon M(D_1) \to H(D)$ be given by the formula $F(f) = f|_{s\in D}$, $f \in M(D_1)$. This function, clearly, is continuous, therefore the lemma is a consequence of Lemma 1 and of the properties of the weak convergence of probability measures, see [1], Theorem 5.1.

Denote

$$S = \{f \in H(D)\colon f(s) \neq 0 \text{ or } f(s) \equiv 0\}.$$

Lemma 3. *The support of the measure $P_{\varphi,D}$ is the set S.*

The proof is given in [3].

For the proof of the theorem we also need the Mergelyan theorem.

Lemma 4. *Let K be a compact subset of $\mathbb{C}$ whose complement is connected. Then any continuous function $f(s)$ on K which is analytic in the interior of K is approximable uniformly on K by polynomials in s.*

The proof can be found, for example, in [5].

Proof of Theorem. First we suppose that $f(s)$ has non-vanishing continuation to $H(D)$. Denote by G the set of functions $g \in H(D)$ such that

$$\sup_{s\in K} |g(s) - f(s)| < \varepsilon.$$

By Lemma 3 the function $f(s)$ is contained in the support S of the random element $\varphi(s,\omega)$. Since by Lemma 2 the probability measure $P_{N,D}$ converges weakly to the measure $P_{\varphi,D}$ as $N \to \infty$ and the set G is open, we deduce from the properties of a weak convergence of probability measures and the support that

$$\liminf_{N\to\infty} \frac{1}{N+1}\#\left\{0 \leqslant m \leqslant N\colon \sup_{s\in K} |\varphi(s+imh) - f(s)| < \varepsilon\right\} \geqslant P_{\varphi,D}(G) > 0.$$

Now let $f(s)$ be as in the statment of the theorem. Then in view of Lemma 4 there exists a sequence $\{p_n(s)\}$ of polynomials such that $p_n(s) \to f(s)$ as $n \to \infty$ uniformly on K. Since $f(s) \neq 0$ on K, we have $p_{n_0}^{(s)} \neq 0$ on K for sufficiently large n_0, and

$$\sup_{s\in K} |f(s) - p_{n_0}(s)| < \frac{\varepsilon}{4}. \tag{5}$$

The polynomial $p_0(s)$ has only finitely many zeros, therefore there exists a region G_1 whose complement is connected such that $K \subset G_1$ and $p_{n_0}(s) \neq 0$ on G_1. Thus there exists a continuous version $\log p_{n_0}(s)$ on G_1 such that $\log p_{n_0}(s)$ is analytic in the interior of G_1. Therefore by Lemma 4 again there exists a sequence $\{q_n(s)\}$ of polynomials such that $q_n(s) \to \log p_{n_0}(s)$ as $n \to \infty$ uniformly on K. Thus, for sufficiently large n_1,

$$\sup_{s\in K} \left|p_{n_0}(s) - e^{q_{n_1}(s)}\right| < \frac{\varepsilon}{4}.$$

Hence and from (5) we obtain

$$\sup_{s\in K} \left|f(s) - e^{q_{n_1}(s)}\right| < \frac{\varepsilon}{2}. \tag{6}$$

From the first part of the proof we deduce that

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \left\{0 \leqslant m \leqslant N\colon \sup_{s\in K} \left|\varphi(s+imh) - e^{q_{n_1}(s)}\right| < \frac{\varepsilon}{2}\right\} > 0. \tag{7}$$

Obviously,

$$\sup_{s\in K} |\varphi(s+imh) - f(s)| \leqslant \sup_{s\in K} \left|\varphi(s+imh) - e^{q_{n_1}(s)}\right| + \sup_{s\in K} \left|f(s) - e^{q_{n_1}(s)}\right|.$$

Therefore by (6)

$$\left\{m\colon \sup_{s\in K} |\varphi(s+imh) - f(s)| < \varepsilon\right\} \supseteq \left\{m\colon \sup_{s\in K} \left|\varphi(s+imh) - e^{q_{n_1}(s)}\right| < \frac{\varepsilon}{2}\right\}.$$

This and (7) yield the assertion of the theorem.

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Diskreti universalumo teorema Matsumoto dzeta funkcijai

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Straipsnyje įrodyta diskreti universalumo teorema Matsumoto dzeta funkcijai.