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On representation formula for solutions of Hamilton–Jacobi equations for unbounded initial conditions

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1. Introduction

In this paper we are concerned with the Cauchy problem

$$L[u] = u_t + H(t, x, u, u_x) = 0, \tag{1}$$
$$u(0, x) = \varphi(x) \tag{2}$$

in $S_T = \{t \in (0, T],\ x \in R^n\}$. It is well known that, in general, there is no hope to find classical solutions of this problem. The first definition of generalized solution for general equation (1) (so called semiconcave solution) were introduced by Douglis [4] and Kruzkov [9]. They proved the existence uniqueness and stability of this solution when Hamiltonian is convex and the initial function is Lipshitz. The work [5] extended these results to the lower semicontinuous and bounded initial function. The other way to investigate problem (1), (2) is to work with the viscosity solutions [1–3, 8]. In these works the general condition of growth of initial function is boundness by linear function (may be from below).

In [6] we consider the equation

$$u_t + (A(x)u_x, u_x) = 0, \tag{3}$$

with matrix $A(x) \in C^{2,\alpha}(R^n)$ satisfying

$$a_1|\xi|^2 \leqslant (A(x)\xi, \xi) \leqslant a_2|\xi|^2, \quad a_1, a_2 \succ 0.$$

We proved the existence and uniqueness of semiconcave solution of (3), (2) satisfying the growth condition

$$\lim_{|x|\to\infty} |x|^{-2}|u(t, x)| = 0,$$

when the initial function is lower semicontinuous and satisfies

$$\lim_{|x|\to\infty} |x|^{-2}|\varphi(x)| = 0.$$

The purpose of this paper is to extend the analogous result for semiconcave solutions of general equation (1). We will prove that the growth of initial function, as $|x| \to \infty$, naturally depends from the growth of $H_v(t,x,u,v)$ as $|u|,|v| \to \infty$. We will give a correct definition of semiconcave solution in this case. The proof of this result is based on representation formula for solutions introduced in [5]. Now we formulate the definition of semiconcave solution and some propositions of this work, which we use further.

2. The representation formula for bounded semiconcave solutions of (1), (2)

Let $H(t,x,u,v) \in C^2(\Omega_T)$, $\Omega_T = \{(t,x) \in S_T,\ u \in R,\ p \in R^n\}$, satisfies:

1. $\exists a \succ 0,\ c_1, c_2 \succ 0, \forall \xi \in R^n, i = 1 \div n$, uniformly in Ω_T:

$$(H_{vv}\xi, \xi) \geqslant a|\xi|^2, \tag{4}$$

$$H(t,x,u,0) = H_v(t,x,u,0) \equiv 0, \tag{5}$$

$$|H_x| \leqslant c_1, \qquad H_u \geqslant -c_2$$

2. $\forall \mu \succ 0, i,j = 1 \div n, \exists N_\mu \succ 0$:

$$H_{uu} \geqslant -\mu,$$

$$|H_{px_i}| + |v_i H_{vu}| \leqslant \mu|p|,$$

$$|H_{x_i x_j}| + |H_{x_j u} p_i| \leqslant \mu|p|^2,$$

in $\Omega_T^\mu = \{(t,x) \in S_T,\ u \in R,\ |p| \geqslant N_\mu\}$.

3. The function $H(t,x,u,v)$ and its derivatives till second order are bounded on $\Omega_T^M = \{(t,x) \in S_T,\ |u| \leqslant M,\ |p| \leqslant M\}, \forall M \succ 0$.

The definition of semiconcave solution of (1), (2), when φ is bounded and Lipshitz, is next [9].

DEFINITION 1. The bounded and Lipshitz on $\bar{S}_T = \{t \in [0,T],\ x \in R^n\}$function $u(t,x)$ is called semiconcave solution of (1), (2) if $u(t,x)$ a.e. in S_T satisfy (1) and there exits the constant $C_\delta \succ 0$ that

$$u(t,x+l) - 2u(t,x) + u(t,x-l) \leqslant C_\delta |l|^2 \tag{6}$$

for $\forall (t,x+l), (t,x), (t,x-l) \in \{t \in [\delta,T],\ x \in R^n\}$.

Let us now suppose φ is bounded and lower semicontinuous in R^n. Then the definition of semiconcave solution is given in [5].

DEFINITION 2. The bounded on $\bar{S}_T$ and locally Lipshitz on S_T function $u(t,x)$ is semiconcave solution of (1), (2) if $u(t,x)$ a.e. in S_T satisfy (1), (6) and

$$\lim_{t \to 0} u(t,x) = \varphi(x). \tag{7}$$

Let $u^{c,m}(t,x,\xi)$ is the semiconcave solution of (1), (2) in the sense of Definition 1 satisfying the initial condition

$$u^{c,m}(0,x,\xi) = \min\{c|x-\xi|, m\}.$$

Suppose

$$G_m(t,x,\xi) = \sup_{c\succ 0} u^{c,m}(t,x,\xi).$$

It is known (Lemma 3, [5]) this function is the semiconcave solution of (1) in the sense of Definition 2 with the initial data

$$G_m(0,x,\xi) = \begin{cases} 0, x=\xi, \\ m, x\neq \xi. \end{cases} \tag{8}$$

The Theorem 2 [5] implies the following

Theorem 1. *Let H satisfies given conditions. Then for bounded and lower semicontinuous initial function $\varphi(x)$ ($|\varphi(x)| \leqslant m$) the formula*

$$u(t,x) = \inf_{\xi\in R^n} (\varphi(\xi) + G_m(t,x,\xi))$$

gives the unique semiconcave solution in the sense of Definition 2.

3. The representation formula for the unbounded semiconcave solution of (1), (2)

Now we will require H_v satisfies

$$|H_v(t,x,u,v)| \leqslant a_1|v|^\alpha + a_2, \tag{9}$$

where $\alpha \geqslant 1$, for some $a_1, a_1 \succ 0$ and all $(t,x) \in S_T$, $u \in R$, $v \in R^n$. We introduce a new definition of semiconcave solution of (1), (2) for the unbounded initial data.

DEFINITION 3. The locally Lipshitz function $u(t,x)$, satisfying a.e. (1) on S_T, is semiconcave solution of (1), (2) if $u(t,x)$ satisfies (7) and

$$u(t,x+l) - 2u(t,x) + u(t,x-l) \leqslant K_n|l|^2 \tag{10}$$

for $\forall (t,x+l), (t,x), (t,x-l) \in S_{n,T} = \{t \in [\frac{1}{n}, T], |x| \leqslant n\}$,

$$\lim \inf_{|x|\to+\infty} |x|^{\frac{\alpha+1}{\alpha}} u(t,x) = 0 \tag{11}$$

uniformly in $[0,T]$.

Define

$$G(t,x,\xi) = \sup_{m\succ 0} G_m(t,x,\xi).$$

Theorem 2. *The function G is the semiconcave solution of* (1), (2) *in the sense of Definition* 3 *satisfying initial condition*

$$G(0,x,\xi) = \begin{cases} 0, & x = \xi, \\ +\infty, & x \neq \xi. \end{cases} \tag{12}$$

Proof. Fix any $\xi \in R^n$. According to (4)

$$0 = L\big[u^{c,m}\big] \geqslant u_t^{c,m} + \frac{a}{2}|u_x^{c,m}|^2 = L^0\big[u^{c,m}\big].$$

We can calculate that

$$G_m^0 = \min\left\{\frac{|x-\xi|^2}{at}, m\right\}$$

gives the semiconcave solution of equation

$$u_t + \frac{a}{2}|u_x|^2 = 0$$

with the initial data (8). So, we have

$$L^0\big[u^{c,m}\big] \leqslant L^0\big[G_m^0\big],$$

and

$$u^{c,m}(0,x,\xi) \leqslant G_m^0(0,x,\xi).$$

Applying the comparison theorem for equation (12) [9] we see

$$u^{c,m}(0,x,\xi) \leqslant G_m^0(0,x,\xi),$$

and so

$$G_m(t,x,\xi) \leqslant \min\left\{\frac{|x-\xi|^2}{at}, m\right\} \leqslant \frac{|x-\xi|^2}{at}.$$

Thus the function G is everywhere finite and satisfies

$$G(t,x,\xi) \leqslant \frac{|x-\xi|^2}{at}. \tag{13}$$

Now we prove that

$$G(t,x,\xi)=G_m(t,x,\xi), \tag{14}$$

when $(t,x)\in O_\xi(m)=\{|x-\xi|\leqslant\sqrt{amt},\ t\in[0,T]\}$. The formula (8) implies $G_s\geqslant G_m$, when $s\geqslant m$. From (5) we observe that constant is solution of (1). Then, using the uniqueness of semiconcave solutions, see Theorem 1, and Lemma 1 [5], we conclude that

$$G_m(t,x,\xi)=\min\{G_s(t,x,\xi),m\}.$$

Letting $s\to+\infty$, we deduce (14). Because the domains $O_\xi(m)$ cover up S_T and G_m is semiconcave solution of (1) we ensure the G is locally Lipshitz, satisfies a.e. in S_T (1) and (10). Notice, that (5) implies $L[u^{c,m}]=L[0]$ and $u^{c,m}(0,x,\xi)\geqslant 0$, so from Theorem 1 we see $u^{c,m}(t,x,\xi)\geqslant 0$. Thus $G\geqslant 0$ and this proves (11).

Finally we prove (12). The Lemma 3 [5] implies that $u^{c,m}=m$, when $(t,x)\in\{|x-\xi|\geqslant\frac{m}{c}+R_{c,m}t\}$, where $\sup_{D_{c,m}}|H_v|\leqslant R_{c,m}$ and $D_{c,m}=\{(t,x)\in S_T,\ |u|\leqslant m,\ |v|\leqslant\sqrt{n}(c+c_1T)e^{c_2T}\}$. From definition of G we have $G\geqslant m$, when $(t,x)\in W_\xi(m)=\{|x-\xi|\geqslant\inf_{c\succ 0}(\frac{m}{c}+R_{c,m}t)=R_m(t)\}$. The (9) and the inequality

$$\left|\frac{a+b}{2}\right|^\alpha\leqslant\frac{|a|^\alpha+|b|^\alpha}{2} \tag{15}$$

for $\alpha\geqslant 1$, $a,b\in R$, imply that we can take $R_{c,m}=\frac{a_1|2\sqrt{n}\,e^{c_2T}c|^\alpha+a_1|2\sqrt{n}\,c_1Te^{c_2T}|^\alpha}{2}+a_2=b_1|c|^\alpha+b_2$, there $b_1=\frac{a_1|2\sqrt{n}\,e^{c_2T}|^\alpha}{2}$ and $b_2=\frac{a_1|2\sqrt{n}\,c_1Te^{c_2T}|^\alpha}{2}+a_2$. Then calculus shows

$$R_m(t)=m^{\frac{\alpha}{\alpha+1}}(b_1t)^{\frac{1}{\alpha+1}}k(a)+b_2t.$$

Let $x\neq\xi$. Then for every large $m\succ 0$ we can find suitably small t that $|x-\xi|\geqslant R_m(t)$ and so $G(t,x,\xi)\geqslant m$. This proves (12) because from (4), (5) follows that solution of (1) is the nondecreasing, as $t\to 0$, function.

Theorem 3. *Let H satisfies given conditions. Then for locally bounded and lower semi-continuous initial function satisfying*

$$\lim_{|x|\to+\infty}\inf|x|^{\frac{\alpha+1}{\alpha}}\varphi(\xi)=0, \tag{16}$$

the formula

$$u(t,x)=\inf_{\xi\in R^n}(\varphi(\xi)+G(t,x,\xi)) \tag{17}$$

gives the unique semiconcave solution in the sense of Definition 3.

Proof. First we will give the more precise estimate from bellow for G. Define

$$\Phi(t,x,\xi)=\begin{cases}0, & |x-\xi|\leqslant b_2t,\\ \frac{1}{(b_1t)^{\frac{1}{\alpha}}k(a)^{\frac{\alpha+1}{\alpha}}}(|x-\xi|-b_2t)^{\frac{\alpha+1}{\alpha}}, & |x-\xi|\geqslant b_2t.\end{cases}\tag{18}$$

The calculus shows that $\Phi(t,x,\xi)=m$ when $(t,x)\in S_m(\xi)=\{|x-\xi|=R_m(t)\}$. We have $G(t,x,\xi)\geqslant m=\Phi(t,x,\xi)$ on $S_m(\xi)$ and $G(t,x,\xi)\geqslant 0=\Phi(t,x,\xi)$, when $|x-\xi|\leqslant b_2t$. The surfaces $S_m(\xi)$, $m\geqslant 0$, and cone $\{|x-\xi|\leqslant b_2t\}$ cover up S_T, so

$$G\geqslant\Phi\tag{19}$$

on S_T. Let $\varphi(x)$ satisfies (16), then

$$\varphi(\xi)+G(t,x,\xi)\geqslant\varphi(\xi)+\Phi(t,x,\xi)\to+\infty,$$

uniformly on compacts in S_T, as $|\xi|\to+\infty$. So, for any compact $K\subset S_T$, we have the ball $B_{r_K}=\{\xi\colon |\xi|\prec r_K\}$ where the infimum in (17) is available when $(t,x)\in K$. Define

$$\varphi_{r_K}(\xi)=\begin{cases}\varphi(\xi), & |\xi|\prec r_K,\\ -M_K, & |\xi|\geqslant r_K,\end{cases}$$

where $M_K=\sup_{|\xi|\prec r_K}|\varphi(\xi)|$. According (14) we can choose $m=m_K$ that $G(t,x,\xi)=G_{m_K}(t,x,\xi)$ if $(t,x)\in K$ and $\xi\in B_{r_K}$. Then

$$u(t,x)=\inf_{\xi\in R^n}(\varphi_{r_K}(\xi)+G_{m_K}(t,x,\xi)),$$

when $(t,x)\in K$. Theorem 1 implies the function u is locally Lipshitz, satisfying (1) a.e. in S_T and (10).

Now we prove (7). Fix any $x\in R^n$. Suppose $\xi(t,x)$ gives infimum in (17), when $t\in[0,T]$. We will prove that $\xi(t,x)\to x$, as $t\to 0$. From (1), (4), (5), (17) we have

$$u(t,x)=\varphi(\xi(t,x))+G(t,x,\xi(t,x))\leqslant\varphi(x).$$

The (19) implies

$$(|x-\xi(t,x)|-b_2t)^{\frac{\alpha+1}{\alpha}}\leqslant(b_1t)^{\frac{1}{\alpha}}k(a)^{\frac{\alpha+1}{\alpha}}(|\varphi(x)|-\varphi(\xi(t,x))).$$

From this inequality and (16) follows the set $\xi(t,x)$ is bounded for all $t\in[0,T]$ and $\xi(t,x)\to x$, as $t\to 0$. Let $|x-\xi(t,x)|\leqslant\varepsilon(t)$, where $\varepsilon(t)\to 0$, as $t\to 0$. Then

$$\inf_{|x-\xi|\leqslant\varepsilon(t)}(\varphi(\xi))\leqslant\inf_{|x-\xi|\leqslant\varepsilon(t)}(\varphi(\xi)+G(t,x,\xi))\leqslant\varphi(x).$$

The semiconcavity of $\varphi(x)$ implies

$$\lim_{t\to 0}\inf_{|x-\xi|\leqslant\varepsilon(t)}(\varphi(\xi))=\varphi(x),$$

and this prove (7).

Now we will show (11). The (16) implies that for any small $\mu\succ 0$ there exists α_μ such that

$$\varphi(x)\geqslant-\alpha_\mu-\mu|x|^{\frac{\alpha+1}{\alpha}}.$$

According (17)–(19) we deduce

$$u(t,x)\geqslant\inf_{\xi\in R^n}\left\{-\alpha_\mu-\mu|\xi|^{\frac{\alpha+1}{\alpha}}+\frac{1}{(b_1t)^{\frac{1}{\alpha}}k(a)^{\frac{\alpha+1}{\alpha}}}(|x-\xi|-b_2t)^{\frac{\alpha+1}{\alpha}}\right\},\qquad(20)$$

if $|x-\xi|\geqslant b_2t$. Otherwise we see

$$u(t,x)\geqslant\inf_{|x-\xi|\leqslant b_2t}\{-\alpha_\mu-\mu|\xi|^{\frac{\alpha+1}{\alpha}}\},$$

and so

$$u(t,x)\geqslant-\alpha_\mu-\mu(|x|+b_2t)^{\frac{\alpha+1}{\alpha}}.$$

Suppose $s=|x-\xi|-b_2t$. Then

$$|\xi|=|x+\xi-x|\leqslant|x|+b_2t+s.$$

Using (15), (20) we can write

$$u(t,x)\geqslant\inf_{s\in[0,+\infty)}\left\{-\alpha_\mu-2^{\frac{1}{\alpha}}\mu(|x|+b_2t)^{\frac{\alpha+1}{\alpha}}+\left(\frac{1}{(b_1t)^{\frac{1}{\alpha}}k(a)^{\frac{\alpha+1}{\alpha}}}-2^{\frac{1}{\alpha}}\mu\right)s^{\frac{\alpha+1}{\alpha}}\right\}.$$

Thus

$$u(t,x)\geqslant-\alpha_\mu-2^{\frac{1}{\alpha}}\mu(|x|+b_2t)^{\frac{\alpha+1}{\alpha}},$$

if $\mu\succ 0$ is sufficiently small. This proves (11).

Now we establish the uniqueness of solution. Let $v(t,x)$ is any other solution. From (11) we have that for any small $\mu\succ 0$ there exists $\beta_\mu\succ 0$ such that

$$v(t,x)\geqslant-\beta_\mu-\mu|x|^{\frac{\alpha+1}{\alpha}}$$

for all $t \in [0, T]$. Define $\delta_n(t, x)$ by formula (17)

$$\delta_n(t, x) = \inf_{\xi \in R^n} \left(\delta_n(0, \xi) + G(t, x, \xi)\right),$$

where

$$\delta_n(0, x) = \begin{cases} +\infty, & |x| \prec n, \\ -\beta_\mu - \mu|n|^{\frac{\alpha+1}{\alpha}}, & |x| \geqslant n. \end{cases}$$

Suppose now

$$v_n(t, x) = \begin{cases} \min\{v(t, x), \delta_n(t, x)\}, & |x| \prec n, \\ -\beta_\mu - \mu|n|^{\frac{\alpha+1}{\alpha}}, & |x| \geqslant n. \end{cases}$$

It is easily to show that $v_n(t, x)$ is bounded semiconcave solution of (1) satisfying the initial condition

$$v_n(0, x) = \begin{cases} \varphi(x), & |x| \prec n, \\ -\beta_\mu - \mu|n|^{\frac{\alpha+1}{\alpha}}, & |x| \geqslant n. \end{cases}$$

According to Theorem 1 it is unique. Now we will prove that for any $(t_0, x_0) \in S_T$ there exists n_0 that

$$\delta_n(t_0, x_0) \geqslant v(t_0, x_0), \tag{21}$$

if $n \geqslant n_0$ and $|x_0| \prec n_0$. Using (19), (20) we see

$$\begin{aligned}\delta_n(t_0, x_0) &\geqslant \min_{|\xi| \succeq n} \left\{ -\beta_\mu - \mu|n|^{\frac{\alpha+1}{\alpha}} + \frac{1}{(b_1 t_0)^{\frac{1}{\alpha}} k(a)^{\frac{\alpha+1}{\alpha}}} (|x_0 - \xi| - b_2 t_0)^{\frac{\alpha+1}{\alpha}} \right\} \\ &\geqslant -\beta_\mu - \mu|n|^{\frac{\alpha+1}{\alpha}} + \frac{1}{(b_1 t_0)^{\frac{1}{\alpha}} k(a)^{\frac{\alpha+1}{\alpha}}} (n - |x_0| - b_2 t_0)^{\frac{\alpha+1}{\alpha}},\end{aligned}$$

when $|x_0| \prec n$, $|x_0| + b_2 t_0 \leqslant n$, so $\lim_{n \to +\infty} \delta_n(t_0, x_0) = +\infty$, if

$$\mu \prec \frac{1}{(b_1 t_0)^{\frac{1}{\alpha}} k(a)^{\frac{\alpha+1}{\alpha}}}.$$

This proves (21) and so $v(t_0, x_0) = v_{n_0}(t_0, x_0)$. The uniqueness of $v_n(t, x)$ ensures the uniqueness of $v(t, x)$.

References

[1] M. Bardi, L.C. Evans, On Hopf's formulas for solutions of Hamilton–Jacobi equations, *Non. Analysis*, **8**(11), 1373–1381 (1984).
[2] E.N. Barron, R. Jensen, Semicontinuous viscosity solutions for Hamilton–Jacobi equations with convex Hamiltonians, *Comm. Partial Diferential Equations*, **15**, 1713–1742 (1990).
[3] M.G. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions, *Bulletin of the Amer. Math. Soc.*, **27**(1), 1–67 (1992).
[4] A. Douglis, Solutions in the large for multi-dimensional nonlinear partial differential equations of first order, *Ann. Inst. Fourier*, Grenoble, **15**(2), 1–35 (1965).
[5] G. Gudynas, Generalized solutions of non-linear equations of first order with the bounded and lower semi-continuous initial function, Preprint, Vilnius, 1–35 (1990) (in Russian).
[6] G. Gudynas, On representation formula for solutions of Hamilton–Jacobi equation for some types of initial conditions, *Mathematical Modelling and Analysis*, **6**(2), 241–250 (2001).
[7] E. Hopf, Generalized solutions of non-linear equations of 1 order, *Journ. of Math. and Mech.*, **14**(6), 951–974 (1965).
[8] H. Ishii, Representation of solutions of of Hamilton–Jacobi equations, *Non. Analysis*, **12**(2), 121–146 (1986).
[9] S.N. Kruzkov, Generalized solutions of the multi-dimensional nonlinear partial differential equations of first order, I, *Math. Sbornik*, **70**(112), 394–415 (1967).
[10] S.N. Kruzkov, Generalized solutions of the multi-dimensional nonlinear partial differential equations of first order, II, *Math. Sbornik*, **72**(114), 108–134 (1967).

Apie Hamiltono–Jakobi lygties sprendinių išraišką neaprėžtos pradinės funkcijos atveju

G. Gudynas

Nagrinėjamas Koši uždavinys bendrai Hamiltono–Jacobi lygčiai

$$u_t + H(t, x, u, u_x) = 0,$$
$$u(0, x) = \varphi(x),$$

kai pradinė funkcija $\varphi(x)$ pusiau tolydi iš apačios ir neaprėžta erdvėje R^n. Iškilaus Hamiltoniano atveju, kai egzistuoja $\alpha \geqslant 1$, $a_1, a_2 \succ 0$ tokie, kad

$$|H_v(t, x, u, v)| \leqslant a_1 t|v|^\alpha + a_2,$$

visiems $(t, x) \in S_T$, $u \in R$, $v \in R^n$, gauta pusiau įgaubto sprendinio išraiška ir įrodyta jo vienatis, kai

$$\lim_{|x|\to+\infty} \inf |x|^{-\frac{\alpha+1}{\alpha}} u(t, x) = 0.$$