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Presheaves on midsymmetrical quantaloids

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1. Introduction

In previous paper [2] we studied presheaves and sheaves on an arbitrary quantaloid (category enriched in the category of complete sup-lattices subject to certain laws), where we had choosed the C.J. Mulvey and M. Nawaz theory [3] as a guideline for futher generalizations taking our inspiration in the matrix approach of G. Van den Bossche [1]. In this note we continue those investigations restricting our interest to a special subclass of quantaloids, "midsymmetrical" quantaloids. In this setting it is possible to express the structural matrix of the underlying Q-set of a regular and separated presheaf in terms of its "restriction" and "diagonal" (Theorem 3.8). We are going to detail it in a subsequent paper.

2. Quantaloids and matrices

We first review some of the basic notions.

DEFINITION 2.1. A *quantaloid* is a locally small category Q such that:

(i) for all u,v objects in Q, the hom-set $Q(u, v)$ is a complete lattice,

(ii) composition of morphisms of Q (in this paper denoted by &) preserves arbitrary joins in both variables:

$$p\& \bigvee_i q_i = \bigvee_i p\&q_i \text{and} (\bigvee_i p_i)\&q = \bigvee_i p_i\&q$$

for all morphisms p,q of Q and for all families (p_i), (q_i) of morphisms of Q (forming respective composable pairs). A quantaloid Q will be called *midsymmetrical* whenever it satisfies

(iii) $p\&(q\&r)\&p' = p\&(r\&q)\&p'$ for all $p \in Q(u, v)$, $q, r \in Q(v, v)$, $p' \in Q(v, v')$.

Note that we use the unconventional left-to-right direction for composition of morphisms. An important example of a one-object midsymmetrical quantaloid is given by the lattice of all closed left-sided (or right-sided) ideals of a non-commutative C^*-algebra.

From now Q will be an arbitrary midsymmetrical quantaloid having a small set of objects. Let Q_0 denote this set and Q_1 the set of morphisms of Q. Let Set/Q_0 denotes the category whose objects are families X of sets X_u indexed by $u \in Q_0$. An element $x \in X_u$ will be called an element over u and we shall sometimes write $d(x)$ for u. Morphisms in Set/Q_0 are families of maps $f_u : X_u \to Y_u$.

DEFINITION 2.2. Let Q be a quantaloid and X and Y be two objects of Set/Q_0. A *matrix* M from X to Y assigns to each pair x, y of $X \times Y$ an element of Q_1:

$$m_{x,y} : d(x) \to d(y).$$

Matrices compose by "matrix multiplication": for $M : X \to Y$, and $N : Y \to Z$, the *composite* $M\&N = L : X \to Z$ has its general element given by

$$l_{x,z} = \bigvee_{y \in Y} m_{x,y} \& n_{y,z}.$$

It is clear that this composition is associative. If Q has units, then the matrix I defined by

$$i_{x,x} = id_{d(x)} \text{ (unit in } Q(d(x), d(x))),$$
$$i_{x,x'} = \bot_{d(x),d(x')} \text{ if } x \neq x' \text{ (bottom element in } Q(d(x), d(x'))),$$

is the unit matrix on X. Matrices from X to Y form a partially ordered set for the pointwise partial order. Sets over Q_0 with matrices as 1-morphisms and 2-morphisms given by partial order determine a bicategory.

3. Q-sets and presheaves on a (midsymmetrical) quantaloid Q

The notions of a (general) Q-set and of a presheaf, which will be given in this section, are taken from [1] and [2].

DEFINITION 3.1. Let Q be a quantaloid. A *Q-set* is an object X of $Sets/Q_0$ provided with a matrix $A : X \to X$ satisfying the following:

Idempotency: $A\&A = A$.

An element $x \in X$ of a Q-set (X, A) will said to be strict provided that it satisfies

Strictness: $a_{x,x} \& a_{x,x'} = a_{x,x'}$ and $a_{x',x} \& a_{x,x} = a_{x',x}$ for all $x' \in X$ and a Q-set (X, A) itself will be called strict whenever every element $x \in X$ is strict.

A Q-set (X, A) will be called regular whenever it satisfies

Regularity: $a_{x,x'} = a_{x,x'} \& a_{x',x} \& a_{x,x'}$ for all $x, x' \in X$.

A Q-set (X, A) will be called separated whenever it satisfies

Separation: if $a_{x,x''} = a_{x',x''}$ and $a_{x'',x} = a_{x'',x'}$ for all $x'' \in X$ (which is just the condition that $a_{x,x} = a_{x',x'} = a_{x,x'} = a_{x',x}$ when x and x' are strict), then $x = x'$.

We shall usually write a_x for $a_{x,x}$. Note that regular Q-sets are strict. An example of a regular Q-set is given in Theorem 3.4 (by (1)).

DEFINITION 3.2. Given a strict Q-set (X, A), the restrictable triplet $(p, x, p^\#)$ (of $Q_1 \times X \times Q_1$) is the element of $Q_1 \times X \times Q_1$ over $d(p, x, p^\#) = dom(p)$ such that $dom(p) = cod(p^\#)$, $cod(p) = dom(p^\#) = d(x)$

$$a_x \& p^\# \& p \& a_x \leqslant a_x, \quad p \& a_x \leqslant p \& p^\# \& p \& a_x, \quad \text{and}$$
$$a_x \& p^\# \leqslant a_x \& p^\# \& p \& p^\#,$$

where the last two inequalities are actually equalities owing to the first inequality.

DEFINITION 3.3. By a *presheaf* on Q will be meant a Q-set (X, A) together with

Restriction: a partial mapping $\upharpoonleft\!\urcorner: Q_1 \times X \times Q_1 \to X$ (more precisely, a "matrix" $\upharpoonleft\!\urcorner = (\upharpoonleft\!\urcorner_{u,v})^{u \in Q_0}_{v \in Q_0}$ of partial mappings $\upharpoonleft\!\urcorner_{u,v}$: $Q(u, v) \times X_v \times Q(v, u) \to X_u$) from the restrictable triplets $(p, x, p^\#) \in Q_1 \times X \times Q_1$ to the elements $p \upharpoonleft x \urcorner p^\#$ of X satisfying the compatibility conditions that:

$$q \upharpoonleft (p \upharpoonleft x \urcorner p^\#) \urcorner q^\# = (q \& p) \upharpoonleft x \urcorner (p^\# \& q^\#), a_x \upharpoonleft x \urcorner a_x = x,$$
$$a_{p \upharpoonleft x \urcorner p^\#, x''} = p \& a_{x,x''} \quad \text{and} \quad a_{x'', p \upharpoonleft x \urcorner p^\#} = a_{x'',x} \& p^\#,$$

for all $x, x'' \in X$ and for all restrictable triplets $(p, x, p^\#), (q, p \upharpoonleft x \urcorner p^\#, q^\#) \in Q_1 \times X \times Q_1$. We call a presheaf $(X, A, \upharpoonleft\!\urcorner)$ on Q regular or separated if the underlying Q-set (X, A) is regular or separated, respectively.

Now we are engaged in an explicit description of the underlying Q-set of a presheaf.

Theorem 3.4. *Given a presheaf $(X, A, \upharpoonleft\!\urcorner)$ on Q, the matrix $A^s = (a^s_{x,x'})^{x \in X}_{x' \in X}$ defined by*

$$a^s_{x,x'} = a_x \& \bigvee \{p \in P_{x,x'} | p \& p^\# \upharpoonleft x \urcorner p \& p^\# = p \upharpoonleft x' \urcorner p^\#\} \& a_{x'} \tag{1}$$

for all $x, x' \in X$, satisfies Idempotency and Regularity (and thus makes (X, A^s) into a regular Q-set), where

$$\begin{aligned} P_{x,x'} = \{ & p \in Q(d(x), d(x')) | \exists p^\# \in Q(d(x'), d(x)) \text{ such that } (p^\#, x, p) \text{ and} \\ & (p, x', p^\#) \text{ restrictable (i.e., } a_x \& p \& p^\# \& a_x \leqslant a_x, \\ & p^\# \& a_x = p^\# \& p \& p^\# \& a_x,\ a_x \& p = a_x \& p \& p^\# \& p,\ a_{x'} \& p^\# \& p \& a_{x'} \leqslant a_{x'}, \\ & p \& a_{x'} = p \& p^\# \& p \& a_{x'}, \text{ and } a_{x'} \& p^\# = a_{x'} \& p^\# \& p \& p^\#)\}. \end{aligned}$$

Moreover, it is compatible with A and $\upharpoonleft\!\urcorner$ in the sense that

$$a^s_{x,x} = a_x, \tag{2}$$
$$a^s_{p \upharpoonleft x \urcorner p^\#, x'} = p \& a^s_{x,x'}, \quad \text{and} \quad a^s_{x', p \upharpoonleft x \urcorner p^\#} = a^s_{x',x} \& p^\#, \tag{3}$$

for all $x, x' \in X$ and all restrictable triplets $(p, x, p^{\#}) \in Q_1 \times X \times Q_1$ (making the triplet $(X, A^s, \upharpoonright\!\upharpoonleft)$ into a regular presheaf on Q).

Proof. Firstly, note that the restrictability of a triplet $(p^{\#}, x, p)$ implies the restrictability of the triplet $(p\&p^{\#}, x, p\&p^{\#})$ (used in (1)). For instance, to verify the inequality $a_x\&p\&p^{\#}\&p\&p^{\#}\&a_x \leqslant a_x$, observe that $a_x\&p\&p^{\#}\&a_x \leqslant a_x$, in view of midsymmetricity, implies that

$$a_x\&p\&p^{\#}\&p\&p^{\#}\&a_x = a_x\&p\&p^{\#}\&a_x\&a_x\&p\&p^{\#}\&a_x \leqslant a_x\&a_x = a_x.$$

Verifying each of the axioms of Definition 3.1 for A^s in turn, we argue as follows:

Idempotency: $a^s_{x,x'} = \bigvee_{x''\in X} a^s_{x,x''}\&a^s_{x'',x'}$ for all $x, x' \in X$, since, on the one hand, we have that

$$\begin{aligned}
&\bigvee_{x''\in X} a_x\&\bigvee\{p \in P_{x,x''} | p\&p^{\#} \upharpoonright x \upharpoonleft p\&p^{\#} = p \upharpoonright x'' \upharpoonleft p^{\#}\}\&a_{x''}\\
&\quad\&\bigvee\{p' \in P_{x'',x'} | p'\&p'^{\#} \upharpoonright x'' \upharpoonleft p'\&p'^{\#} = p' \upharpoonright x' \upharpoonleft p'^{\#}\}\&a_{x'}\\
&\leqslant \bigvee_{x''\in X} a_x\&\bigvee\{p\&a_{x''}\&p' \in P_{x,x'} | (p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#}) \upharpoonright x \upharpoonleft (p\&\\
&\quad a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#}) = (p\&a_{x''}\&p') \upharpoonright x' \upharpoonleft (p'^{\#}\&a_{x''}\&p^{\#})\}\&a_{x'}\\
&\leqslant a_x \bigvee\{p'' \in P_{x,x'} | p''\&p''^{\#} \upharpoonright x \upharpoonleft p''\&p''^{\#} = p'' \upharpoonright x' \upharpoonleft p''^{\#}\}\&a_{x'},
\end{aligned}$$

because, from the midsymmetricity of the quantaloid, one deduce that

$$\begin{aligned}
&\{p \in P_{x,x''}, p\&p^{\#} \upharpoonright x \upharpoonleft p\&p^{\#} = p \upharpoonright x'' \upharpoonleft p^{\#}, p' \in P_{x'',x'}, \quad \text{and}\\
&p'\&p'^{\#} \upharpoonright x'' \upharpoonleft p'\&p'^{\#} = p' \upharpoonright x' \upharpoonleft p'^{\#}\} \Rightarrow p\&a_{x''}\&p' \in P_{x,x'}
\end{aligned}$$

(for instance, to verify the inequality:

$$a_x\&(p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#})\&a_x \leqslant a_x,$$

observe that relations $a_x\&p\&p^{\#}\&a_x \leqslant a_x$, $a_{x''}\&p'\&p'^{\#}\&ax'' \leqslant a_{x''}$, and $p\&p^{\#}\&a_x\&p\&p^{\#} = p\&a_{x''}\&p^{\#}$, imply that

$$\begin{aligned}
a_x\&p\&(a_{x''}\&p'\&p'^{\#}\&a_{x''})\&p^{\#}\&a_x &\leqslant a_x\&(p\&a_{x''}\&p^{\#})\&a_x\\
&= a_x\&p\&p^{\#}\&a_x\&p\&p^{\#}\&a_x \leqslant a_x;
\end{aligned}$$

while, for

$$a_x\&(p\&a_{x''}\&p') = a_x\&(p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#})\&(p\&a_{x''}\&p'),$$

we calculate as follows:

$$\begin{aligned}
&a_x\&p\&a_{x''}\&(p'\&p'^{\#})\&a_{x''}\&(p^{\#}\&p)\&a_{x''}\&p'\\
&\quad = (a_x\&p\&p^{\#}\&p)\&(a_{x''}\&p'\&p'^{\#}\&p') = a_x\&p\&a_{x''}\&p')
\end{aligned}$$

and because

$$\begin{aligned}
&\{p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x'' \Lsh p^{\#} \text{ and } p'\&p'^{\#} \Rsh x'' \Lsh p'\&p'^{\#} = p' \Rsh x' \Lsh p'^{\#}\}\\
&\Rightarrow \{p\&a_{x''}\&p'\&p'^{\#}\&a_{x''}\&p^{\#} \Rsh p\&p^{\#} \Rsh x \Lsh p\&p^{\#} \Lsh p\&a_{x''}\&p'\&p'^{\#}\&a_{x''}\&p^{\#}\\
&= p\&a_{x''}\&p'\&p'^{\#}\&a_{x''}\&p^{\#} \Rsh p \Rsh x'' \Lsh p^{\#} \Lsh p\&a_{x''}\&p'\&p'^{\#}\&a_{x''}\&p^{\#} \quad \text{and}\\
&p\&a_{x''} \Rsh p'\&p'^{\#} \Rsh x'' \Lsh p'\&p'^{\#} \Lsh a_{x''}\&p^{\#} = p\&a_{x''} \Rsh p' \Rsh x' \Lsh p'^{\#} \Lsh a_{x''}\&p^{\#}\}\\
&\Rightarrow \{p\&a_{x''}\&p'\&p'^{\#}\&(a_{x''}\&p^{\#}\&p\&p^{\#}) \Rsh x \Lsh (p\&p^{\#}\&p\&a_{x''})\&p'\&p'^{\#}\&a_{x''}\&p^{\#}\\
&= p\&a_{x''}\&(p'\&p'^{\#})\&a_{x''}\&(p^{\#}\&p)\&a_{x''} \Rsh x'' \Lsh a_{x''}\&(p^{\#}\&p)\&a_{x''}\&(p'\&p'^{\#})\&a_{x''}\&p^{\#}\\
&\text{and} \quad p\&a_{x''}\&p'\&p'^{\#} \Rsh x'' \Lsh p'\&p'^{\#}\&a_{x''}\&p^{\#} = p\&a_{x''}\&p' \Rsh x' \Lsh p'^{\#}\&a_{x''}\&p^{\#}\}\\
&\Rightarrow \{p\&a_{x''}\&p'\&p'^{\#}\&a_{x''}\&p^{\#} \Rsh x \Lsh p\&a_{x''}\&p'\&p'^{\#}\&a_{x''}\&p^{\#}\\
&= (p\&p^{\#}\&p\&a_{x''})\&p'\&p'^{\#}\&a_{x''} \Rsh x'' \Lsh a_{x''}\&p'\&p'^{\#}\&(a_{x''}\&p^{\#}\&p\&p^{\#})\\
&\text{and} \quad p\&a_{x''}\&p'\&p'^{\#} \Rsh x'' \Lsh p'\&p'^{\#}\&a_{x''}\&p^{\#} = p\&p' \Rsh x' \Lsh p'^{\#}\&a_{x''}\&p^{\#}\}\\
&\Rightarrow (p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#}) \Rsh x \Lsh (p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#})\\
&= p\&a_{x''}\&p'\&p'^{\#} \Rsh x'' \Lsh p'\&p'^{\#}\&a_{x''}\&p^{\#} = (p\&a_{x''}\&p') \Rsh x' \Lsh (p'^{\#}\&a_{x''}\&p^{\#})\\
&\Rightarrow (p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#}) \Rsh x \Lsh (p\&a_{x''}\&p')\&(p'^{\#}\&a_{x''}\&p^{\#})\\
&= (p\&a_{x''}\&p') \Rsh x' \Lsh (p'^{\#}\&a_{x''}\&p^{\#}),
\end{aligned}$$

while, on the other hand,

$$\begin{aligned}
&\bigvee_{x''\in X} a_x\&\bigvee\{p \in P_{x,x''} | p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x'' \Lsh p^{\#}\}\&a_{x''}\\
&\quad \&a_{x''}\&\bigvee\{p' \in P_{x'',x'} | p'\&p'^{\#} \Rsh x'' \Lsh p'\&p'^{\#} = p' \Rsh x' \Lsh p'^{\#}\}\&a_{x'}\\
&\qquad \geqslant a_x\&\bigvee\{p \in P_{x,x'} | p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x' \Lsh p^{\#}\}\&a_{x'}
\end{aligned}$$

(putting $x'' = x'$ and $p' = a_{x'}$);

Regularity: $a^s_{x,x'} = a^s_{x,x'}\&a^s_{x',x}\&a^s_{x,x'}$ for all $x, x' \in X$, since, in the one direction, we have that

$$a^s_{x,x'}\&a^s_{x',x}\&a^s_{x,x'} \leqslant a^s_{x,x'} \text{ (by Idempotency of } A^s \text{ twice)},$$

while, in the other,

$$\begin{aligned}
&a_x\&\bigvee\{p \in P_{x,x'} | p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x' \Lsh p^{\#}\}\&a_{x'}\\
&\&a_{x'}\&\bigvee\{q \in P_{x',x} | q\&q^{\#} \Rsh x' \Lsh q\&q^{\#} = q \Rsh x \Lsh q^{\#}\}\&a_x\\
&\&a_x\&\bigvee\{r \in P_{x,x'} | r\&r^{\#} \Rsh x \Lsh r\&r^{\#} = r \Rsh x' \Lsh r^{\#}\}\&a_{x'}\\
&\geqslant a_x\&\bigvee\{p\&p^{\#}\&p\&a_{x'} (= p\&a_{x'}) | p \in P_{x,x'}, p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x' \Lsh p^{\#}\}\&a_{x'}
\end{aligned}$$

(putting $q = r^{\#} = p^{\#}$ and $q^{\#} = r = p$ and noting that

$$p^{\#}\&p \Rsh x' \Lsh p^{\#}\&p = p^{\#} \Rsh x \Lsh p \Leftrightarrow p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x' \Lsh p^{\#}$$

for all $p \in P_{x,x'}$), which completes the verification that the underlying family X of sets of the presheaf $(X, A, \Rsh\Lsh)$ together with A^s is a Q-set.

For (2), in the one direction, we have that

$$a_x = a_x \& a_x \& a_x \leqslant a_x \& \bigvee\{p \in P_{x,x} | p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x' \Lsh p^{\#}\}\&a_x,$$

while, in the other,

$$a_x \& \bigvee\{p \in P_{x,x} | p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x' \Lsh p^{\#}\}\&a_x \leqslant a_x,$$

since from the relation $p\&p^{\#} \Rsh x \Lsh p\&p^{\#} = p \Rsh x \Lsh p^{\#}$, it folows that

$$\begin{aligned} a_x\&p\&a_x &= a_x\&a_{p\Rsh x\Lsh p^{\#},x} = a_x\&a_{p\&p^{\#}\Rsh x\Lsh p\&p^{\#},x} \\ &= a_x\&p\&p^{\#}\&a_x \leqslant a_x \ \text{(by the restrictability of } (p, x, p^{\#})). \end{aligned}$$

To prove the first relation in (3), by the observation that

$$q \in P_{p\Rsh x\Lsh p^{\#},x'} \Rightarrow a_x\&p^{\#}\&q\&a_{x'} \in P_{x,x'}$$

and that

$$\begin{aligned} & q\&q^{\#} \Rsh p \Rsh x \Lsh p^{\#} \Lsh q\&q^{\#} = q \Rsh x' \Lsh q^{\#} \\ & \Rightarrow a_x\&p^{\#}\&q\&a_{x'}\&q^{\#} \Rsh q\&q^{\#} \Rsh p \Rsh x \Lsh p^{\#} \Lsh q\&q^{\#} \Lsh q\&a_{x'}\&q^{\#}\&p\&a_x \\ & = a_x\&p^{\#}\&q\&a_{x'}\&q^{\#} \Rsh q \Rsh x' \Lsh q^{\#} \Lsh q\&a_{x'}\&q^{\#}\&p\&a_x \\ & \Rightarrow a_x\&p^{\#}\&q\&(a_{x'}\&q^{\#}\&q\&q^{\#})\&p \Rsh x \Lsh p^{\#}\&(q\&q^{\#}\&q\&a_{x'})\&q^{\#}\&p\&a_x \\ & = a_x\&p^{\#}\&(q\&a_{x'}\&q^{\#}\&q\&a_{x'}) \Rsh x' \Lsh (a_{x'}\&q^{\#}\&q\&a_{x'}\&q^{\#})\&p\&a_x \\ & \Rightarrow (a_x\&p^{\#}\&q\&a_{x'})\&(a_{x'}\&q^{\#}\&p\&a_x) \Rsh x \Lsh (a_x\&p^{\#}\&q\&a_{x'}) \\ & \quad \&(a_{x'}\&q^{\#}\&p\&a_x) = (a_x\&p^{\#}\&q\&a_{x'}) \Rsh x' \Lsh (a_{x'}\&q^{\#}\&p\&a_x), \end{aligned}$$

we obtain the following estimations

$$\begin{aligned} & a^s_{p\Rsh x\Lsh p^{\#},x'} = a_{p\Rsh x\Lsh p^{\#}} \& \bigvee\{q \in P_{p\Rsh x\Lsh p^{\#},x'} | q\&q^{\#} \Rsh p \Rsh x \Lsh p^{\#} \Lsh q\&q^{\#} \\ & = q \Rsh x' \Lsh q^{\#}\}\&a_{x'} \leqslant p\& \bigvee\{a_x\&p^{\#}\&q\&a_{x'} \in P_{x,x'} | (a_x\&p^{\#}\&q\&a_{x'}) \\ & \quad \&(a_{x'}\&q^{\#}\&p\&e_x) \Rsh x \Lsh (a_x\&p^{\#}\&q\&a_{x'})\&(a_{x'}\&q^{\#}\&p\&a_x) \\ & = (a_x\&p^{\#}\&q\&a_{x'}) \Rsh x' \Lsh (a_{x'}\&q^{\#}\&p\&a_x))\} \\ & \leqslant p\& \bigvee\{q' \in P_{x,x'} | q'\&q'^{\#} \Rsh x \Lsh q'\&q'^{\#} = q' \Rsh x' \Lsh q'^{\#}\}, \end{aligned}$$

which implies that

$$a^s_{p\Rsh x\Lsh p^{\#},x'} \leqslant p\&a^s_{x,x'}.$$

For the converse, using the implication

$$\{(p,x,p^{\#}) \text{ is restrictable, } r\in P_{x,x'}, \text{ and } r\&r^{\#}\Rsh x\Lsh r\&r^{\#}=r\Rsh x'\Lsh r^{\#}$$
$$\text{(from which one has } r\&a_{x'}=r\&r^{\#}\&a_{x,x'})\}\Rightarrow p\&a_x\&r\in P_{p\Rsh x\Lsh p^{\#},x'}$$

and the following series of implications

$$\begin{aligned}
&\{(p,x,p^{\#}) \text{ is restrictable and } r\&r^{\#}\Rsh x\Lsh r\&r^{\#}=r\Rsh x'\Lsh r^{\#}\}\\
&\Rightarrow p\&a_x\&r\&r^{\#}\Rsh r\&r^{\#}\Rsh x\Lsh r\&r^{\#}\Lsh r\&r^{\#}\&a_x\&p^{\#}\\
&=p\&a_x\&r\&r^{\#}\Rsh r\Rsh x'\Lsh r^{\#}\Lsh r\&r^{\#}\&a_x\&p^{\#}\\
&\Rightarrow p\&(a_x\&r\&r^{\#}\&r)\&r^{\#}\Rsh x\Lsh r\&(r^{\#}\&r\&r^{\#}\&a_x)\&p^{\#}\\
&=p\&(a_x\&r\&r^{\#}\&r)\Rsh x'\Lsh(r^{\#}\&r\&r^{\#}\&a_x)\&p^{\#}\\
&\Rightarrow p\&(p^{\#}\&p)\&a_x\&(r\&r^{\#})\&a_x\Rsh x\Lsh a_x\&(r\&r^{\#})\\
&\quad\&a_x\&(p^{\#}\&p)\&p^{\#}=(p\&a_x\&r)\Rsh x'\Lsh(r^{\#}\&a_x\&p^{\#})\\
&\Rightarrow(p\&a_x\&r)\&(r^{\#}\&a_x\&p^{\#})\&(p\&a_x\Rsh x\Lsh a_x\&p^{\#})\\
&\&(p\&a_x\&r)\&(r^{\#}\&a_x\&p^{\#})=(p\&a_x\&r)\Rsh x'\Lsh(r^{\#}\&a_x\&p^{\#})\\
&\Rightarrow(p\&a_x\&r)\&(r^{\#}\&a_x\&p^{\#})\Rsh(p\Rsh x\Lsh p^{\#})\Lsh(p\&a_x\&r)\\
&\quad\&(r^{\#}\&a_x\&p^{\#})=(p\&a_x\&r)\Rsh x'\Lsh(r^{\#}\&a_x\&p^{\#}),
\end{aligned}$$

we calculate

$$\begin{aligned}
&p\&a^s_{x,x'}=p\&a_x\&\bigvee\{r\in P_{x,x'}\,|\,r\&r^{\#}\Rsh x\Lsh r\&r^{\#}=r\Rsh x'\Lsh r^{\#}\}\&a_{x'}\\
&=p\&(p^{\#}\&p)\&a_x\&\bigvee\{a_x\&r\,|\,r\in P_{x,x'},r\&r^{\#}\Rsh x\Lsh r\&r^{\#}=r\Rsh x'\Lsh r^{\#}\}\&a_{x'}\\
&=(p\&a_x\&p^{\#})\&\bigvee\{p\&a_x\&r\,|\,r\in P_{x,x'},r\&r^{\#}\Rsh x\Lsh r\&r^{\#}=r\Rsh x'\Lsh r^{\#}\}\&a_{x'}\\
&\leqslant a_{p\Rsh x\Lsh p^{\#}}\&\bigvee\{p\&a_x\&r\in P_{p\Rsh x\Lsh p^{\#},x'}\,|\,(p\&a_x\&r)\&(r^{\#}\&a_x\&p^{\#})\Rsh(p\Rsh x\Lsh p^{\#})\Lsh\\
&\quad(p\&a_x\&r)\&(r^{\#}\&a_x\&p^{\#})=(p\&a_x\&r)\Rsh x'\Lsh(r^{\#}\&a_x\&p^{\#})\}\&a_{x'}\\
&\leqslant a_{p\Rsh x\Lsh p^{\#}}\&\bigvee\{r'\in P_{p\Rsh x\Lsh p^{\#},x'}\,|\,r'\&r'^{\#}\Rsh(p\Rsh x\Lsh p^{\#})\Lsh r'\&r'^{\#}=r'\Rsh x'\Lsh r'^{\#}\}\&a_{x'}\\
&=a^s_{p\Rsh x\Lsh p^{\#},x'},
\end{aligned}$$

which completes the proof of the first relation in (3), while the verification of the rest is similar.

We obtain further properties of the regular Q-set (X,A^s) (introduced in Theorem 3.4) in the next few propositions.

Lemma 3.5. *Let* $(X,A,\Rsh\Lsh)$ *be a presheaf on* Q *and* $A^s=(a^s_{x,x'})^{x\in X}_{x'\in X}$ *be the structural matrix defined by* (1) *(in Theorem* 3.4*). Then the inequality*

$$a^s_{x,x'}\leqslant a_{x,x'} \tag{4}$$

holds for all $x, x' \in X$ *,i.e.,* $A^s \leqslant A$.

Proof. Let us consider elements $x, x' \in X$ and $p, p^\# \in Q_1$ with the properties that $(p^\#, x, p)$ and $(p, x', p^\#)$ are restrictable (i.e., $p \in P_{x,x'}$) and that the equality $p\&p^\# \Rsh x \Lsh p\&p^\# = p \Rsh x' \Lsh p^\#$ holds. Then we infer from (3) that $p\&p^\#\&a_{x,x'} = p\&a_{x'}$. Now we obtain

$$a_x\&p\&a_{x'} = a_x\&p\&p^\#\&a_{x,x'} = (a_x\&p\&p^\#\&a_x)\&a_{x,x'} \leqslant a_x\&a_{x,x'} = a_{x,x'},$$

whence (4).

Lemma 3.6. *Let* $(X, A, \Rsh\Lsh)$ *be a separated presheaf on* Q *(i.e.,* (X, A) *be a separated* Q*-set). Let* $(p, x', p^\#)$ *and* $(p^\#, x, p)$ *be restrictable triplets. Then the relation*

$$p\&p^\# \Rsh x \Lsh p\&p^\# = p \Rsh x' \Lsh p^\# \tag{5}$$

(presented in (1)*) is just the condition that*

$$p\&p^\#\&a_x\&p\&p^\# = p\&a_{x'}\&p^\# = p\&p^\#\&a_{x,x'}\&p^\# = p\&a_{x',x}\&p\&p^\#. \tag{6}$$

Proof. To prove the implication from (5) to (6), assume (5). Then

$$a_{p\&p^\# \Rsh x \Lsh p\&p^\#} = a_{p \Rsh x' \Lsh p^\#} = a_{p\&p^\# \Rsh x \Lsh p\&p^\#, p \Rsh x' \Lsh p^\#} = a_{p \Rsh x' \Lsh p^\#, p\&p^\# \Rsh x \Lsh p\&p^\#}, \tag{7}$$

which implies (6). Assume now (6). Then it is clear that the relations (7) hold. By Separation, this means that (5) holds.

Lemma 3.7. *If a presheaf* $(X, A, \Rsh\Lsh)$ *is separated, then the triplets* $(a_{x',x}, x, a_{x,x'})$ *and* $(a_{x,x'}, x', a_{x',x})$ *are restrictable (i.e.,* $a_{x,x'} \in P_{x,x'}$ *) and*

$$a_{x,x'}\&a_{x',x} \Rsh x \Lsh a_{x,x'}\&a_{x',x} = a_{x,x'} \Rsh x' \Lsh a_{x',x}$$

for all $x, x' \in X$.

Proof. In view of Idempotency and Regularity of A , it is clear that $a_{x,x'} \in P_{x,x'}$ for all $x, x' \in X$. Moreover, we have that

$$a_{x,x'}\&a_{x',x}\&a_x\&a_{x,x'}\&a_{x',x} = a_{x,x'}\&a_{x'}\&a_{x',x} = a_{x,x'}\&a_{x',x}\&a_{x,x'}\&a_{x',x}$$

for all $x, x' \in X$. By Lemma 3.6, this just says the statement is correct.

Theorem 3.8. *If* $(X, A, \Rsh\Lsh)$ *is a regular and separated presheaf, then the* Q*-set* (X, A^s) *introduced in Theorem* 3.4 *(by* (1)*) is exactly the underlying* Q*-set* (X, A) *, that is,*

$$a_{x,x'} = a_x\& \bigvee \{p \in P_{x,x'} | p\&p^\# \Rsh x \Lsh p\&p^\# = p \Rsh x' \Lsh p^\#\}\&a_{x'}$$

for all $x, x' \in X$.

Proof. From Lemma 3.7, it follows that

$$a_{x,x'} = a_x \& a_{x,x'} \& a_{x'} \leqslant a_x \& \bigvee \{p \in P_{x,x'} | p\&p^{\#} \upharpoonright x \urcorner p\&p^{\#} = p \upharpoonright x' \urcorner p^{\#}\} \& a_{x'}$$

for all $x, x' \in X$. In view of Lemma 3.5, this concludes the proof.

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Priešpluoštai virš midsimetriškų kvantaloidų

R.P. Gylys

Ankstesniame straipsnyje [2] mes nagrinėjome priešpluoščius ir pluoštus virš laisvojo kvantaloido. Šiame darbe tiriami priešpluoščiai virš midsimetriško kvantaloido. Išvesta priešpluoščio struktūrinės matricos išraiška per jo siaurinį ir diagonalę.