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A multidimensional discrete limit theorem for the Matsumoto zeta-function in the space of analytic functions

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1. Introduction

Let $s=\sigma+it$ be a complex variable, and let $\mathbb{N}$ and $\mathbb{C}$ denote the set of all natural numbers and complex numbers, respectively. The Matsumoto zeta-function $\varphi(s)$ is defined by

$$\varphi(s)=\prod_{m=1}^{\infty} A_m^{-1}(p_m^{-s}). \tag{1}$$

Here

$$A_m(x)=\prod_{j=1}^{g(m)}\left(1-a_m^{(j)}x^{f(j,m)}\right)$$

is a polynomial of degree $f(1,m)+\dots f(g(m),m)$, where $a_m^{(j)}$ are complex numbers, $g(m)$ and $f(j,m)$ are natural numbers, $1\leqslant j\leqslant g(m)$, $m\in\mathbb{N}$, and p_m denotes the m-th prime number. In [6] K. Matsumoto assumed the conditions

$$g(m)=Bp_m^{\alpha},\quad |a_m^{(j)}|\leqslant p_m^{\beta}, \tag{2}$$

where B is a quantity bounded by a constant, and α and β are non-negative constants. Under these conditions infinitedimensional product (1) converges absolutely in the half-plane $\sigma>\alpha+\beta+1$, and defines a holomorphic function with no zeros.

The discrete value-distribution for the Matsumoto zeta-function was investigated in [2], [3], [4].

The aim of this note is to prove a multidimensional discrete limit theorem in the sence of the weak convergence of probability measures for the Matsumoto zeta-functions in the space of analytic functions.

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Let $A_{lm}(x)$ be a polynomial (respectively functions $\varphi_l(s)$) given by

$$A_{lm}(x) = \prod_{j=1}^{g_l(m)} \left(1 - a_{lm}^{(j)} x^{f_l(j,m)}\right), \quad l = 1, \ldots, r,$$

where $r \geqslant 2$. Suppose the functions $\varphi_l(s)$ are analytic in the strip $D = \{s \in \mathbb{C} : \rho_0 < \sigma < \alpha + \beta + 1\}$, where $\alpha + \beta + \frac{1}{2} < \rho_0 < \alpha + \beta + 1$, and conditions (2) are satisfied. Moreover, for $\sigma > \rho_0$

$$\varphi_l(\sigma + it) = B|t|^{\delta}, \tag{3}$$

with some positive δ, and

$$\int_0^T |\varphi_l(\sigma + it)|^2 \mathrm{d}t = BT, \quad T \to \infty. \tag{4}$$

Denote by $H(D)$ the space of analytic on D functions equipped with the topology of uniform convergence on compacta, and by $H^r(D)$ denote the Cartesian product of $\underbrace{H(D) \times \ldots \times H(D)}_{r}$.

Let γ be the unit circle on $\mathbb{C}$, i.e., $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and let

$$\Omega = \prod_p \gamma_p,$$

$\gamma_p = \gamma$ for all primes p. With the product topology and pointwise multiplication the infinite-dimensional torus Ω is a compact topological Abelian group. Let m_H be the probability Haar measure on $(\Omega, \mathcal{B}(\Omega))$, where $\mathcal{B}(S)$ denotes the class of Borel sets of the space S. This gives a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Denote by $\omega(p)$ the projection of $\omega \in \Omega$ to the coordinate space γ_p. Then on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ an $H(D)$-valued random element is defined by

$$\varphi_l(s, \omega) = \prod_{m=1}^{\infty} \prod_{j=1}^{g_l(m)} \left(1 - \frac{a_{lm}^{(j)} \omega^{f_l(j,m)}(p_m)}{p_m^{s f_l(j,m)}}\right)^{-1}, \quad l = 1, \ldots, r.$$

On $(\Omega, \mathcal{B}(\Omega), m_H)$ define an $H^r(D)$-valued random element

$$\Phi(s, \omega) = \big(\varphi_1(s, \omega), \ldots, \varphi_r(s, \omega)\big), \quad \omega \in \Omega, \quad s \in D.$$

Moreover, let

$$\Phi(s) = \big(\varphi_1(s), \ldots, \varphi_r(s)\big),$$

and define a probability measure

$$P_N(A) = \mu_N\big(\Phi(s+ikh) \in A\big), \quad A \in \mathcal{B}\big(H^r(D)\big).$$

Here

$$\mu_N(\ldots) = \frac{1}{N+1}\#\{0 \leqslant k \leqslant N, \ldots\},$$

h is a real fixed positive number, $N \in \mathbb{N}$ and instead of dots a condition satisfied by k is to be written. Let P_Φ stand for the distribution of the random element $\Phi(s,\omega)$, i.e.,

$$P_\Phi(A) = m_H\big(\omega \in \Omega : \Phi(s,\omega) \in A\big), \quad A \in \mathcal{B}\big(H^r(D)\big).$$

Theorem. *Suppose that* $\exp\left\{\frac{2\pi k}{h}\right\}$ *is irrational for all integers* $k \neq 0$, *and the functions* $\varphi_l(s)$, $l = 1,\ldots,r$, *satisfied the conditions* (3), (4). *Then the probability measure* P_N *converges weakly to* P_Φ *as* $N \to \infty$.

2. Proof of the Theorem

First we recall that the family of probability measures $\{P\}$ is relatively compact if every sequence of elements of $\{P\}$ contains a weakly convergent subsequence. The family $\{P\}$ is tight if for an arbitrary $\varepsilon > 0$ there exists a compact set K such that $P(K) > 1 - \varepsilon$ for all $P \in \{P\}$.

Lemma 1. *The family of probability measures* $\{P_N\}$ *is relatively compact.*

Proof. Let $P_{\varphi l}$ is the distribution of the random element $\varphi_l(s,\omega)$, $l = 1,\ldots,r$. Then by the Theorem of [3] the probability measure

$$P_{l,N}(A) = \mu_N\big(\varphi_l(s+ikh) \in A\big), \quad A \in \mathcal{B}\big(H(D)\big)$$

converges weakly to $P_{\varphi,l}$ as $N \to \infty$, $l = 1,\ldots,r$. Hence we have that the family of probability measures $\{P_{l,N}\}$ is relatively compact, $l = 1,\ldots,r$. Since $H(D)$ is a complete separable space, then we obtain by Prokhorov's theorem from [1] that the family $\{P_{l,N}\}$ is tight. This means that for every $\varepsilon > 0$ there exists a compact set $K_l \subset H(D)$ such that

$$P_{l,N}\big(H(D) \setminus K_l\big) < \frac{\varepsilon}{r}, \quad l = 1,\ldots,r. \tag{5}$$

Denote by θ a random variable defined on probability space $\big(\Omega_0, \mathcal{B}(\Omega_0), \mathbb{P}\big)$ such that

$$\mathbb{P}(\theta_N = kh) = \frac{1}{N+1}, \quad k = 0,1,\ldots,N,$$

where $N \in \mathbb{N}$, and let the $H(D)$-valued random element $\Phi_N(s)$ be given by

$$\Phi_N(s) = \big(\varphi_{1,N}(s), \dots, \varphi_{r,N}(s)\big).$$

Here

$$\varphi_{l,N}(s) = \varphi_l(s + i\theta_N), \quad l = 1, \dots, r.$$

Then by the definition of $P_{l,N}$ and (5) we have

$$\mathbb{P}\big(\varphi_{l,N} \in H(D) \setminus K_l\big) < \frac{\varepsilon}{r}, \quad l = 1, \dots, r. \tag{6}$$

Now let $K = K_1 \times \dots \times K_r$. Then K is a compact set of the space $H^r(D)$, and, by (6), we obtain

$$\begin{aligned} P_N\big(H^r(D) \setminus K\big) &= \mathbb{P}\big(\Phi_N(s) \in H^r(D) \setminus K\big) = \mathbb{P}\Big(\bigcup_{l=1}^{r} \big(\varphi_{l,N}(s) \in H(D) \setminus K_l\big)\Big) \\ &\leqslant \sum_{l=1}^{r} \mathbb{P}\big(\varphi_{l,N}(s) \in H(D) \setminus K_l\big) < \varepsilon. \end{aligned} \tag{7}$$

(7) show that the family $\{P_N\}$ is tight. By the Prokhorov theorem [1] it is relatively compact. The lemma is proved.

Now let $s_1, \dots, s_n$ be arbitrary points on D, and

$$\sigma_1 = \min_{1 \leqslant l \leqslant n} \Re s_l.$$

Then $\sigma_2 = \rho_0 - \sigma_1 < 0$, and we set

$$\hat{D} = \{s \in \mathbb{C} : \sigma > \sigma_2\}.$$

Moreover, let u_{lm} be arbitrary complex numbers, $1 \leqslant l \leqslant r$, $1 \leqslant m \leqslant n$. Define a function $u : H^r(D) \to H(\hat{D})$ by the formula

$$u(\varphi_1, \dots, \varphi_n) = \sum_{l=1}^{r} \sum_{m=1}^{n} u_{lm} \varphi_l(s_m + s),$$

where $s \in \hat{D}$, $\varphi_l \in H(D)$, $l = 1, \dots, r$. Let

$$W(s) = u\big(\varphi_1(s), \dots, \varphi_r(s)\big),$$

and denote by $\xrightarrow[N \to \infty]{\mathcal{D}}$ the convergence in distribution.

Lemma 2. *We have*

$$W(s+i\theta_N)\xrightarrow[N\to\infty]{\mathcal{D}}u(\Phi(s)).$$

Proof. The proof is similar to that Lemma 14 of [4]. By Lemma 1 there exists a sequence $N_1\to\infty$ such that the measure P_{N_1} converges weakly to some probability measure P on $\big(H^r(D),\mathcal{B}(H^r(D))\big)$ as $N_1\to\infty$. Let P be the distribution $H^r(D)$-valued random element

$$\Phi_1(s)=\big(\varphi_{11}(s),\dots,\varphi_{1r}(s)\big).$$

Then

$$\Phi_{N_1}\xrightarrow[N_1\to\infty]{\mathcal{D}}\Phi_1. \tag{8}$$

The function u is continuous. Consequently

$$u(\Phi_{N_1})\xrightarrow[N_1\to\infty]{\mathcal{D}}u(\Phi_1).$$

Therefore, by definition of W, we find

$$W(s+i\theta_{N_1})\xrightarrow[N_1\to\infty]{\mathcal{D}}u(\Phi_1). \tag{9}$$

For $\sigma>\alpha+\beta+\frac{1}{2}$ by the definition of the function u

$$W(s)=\sum_{l=1}^{r}\sum_{m=1}^{n}u_{lm}\varphi_l(s_m+s)=\sum_{k=1}^{\infty}\frac{a_k}{k^s}, \tag{10}$$

where

$$a_k=\sum_{l=1}^{r}\sum_{m=1}^{n}\frac{u_{lm}c_l(k)}{k^{s_m}},$$

since in this region $\varphi_l(s)$ is presented by an absolutely convergent Dirichlet series

$$\varphi_l(s)=\sum_{k=1}^{\infty}\frac{c_k(k)}{k^s},\quad l=1,\dots,r.$$

Moreover, $c_l(k)=Bk^{\alpha+\beta+\varepsilon}$ for every $\varepsilon>0$. The function $W(s)$ satisfies the same conditions as the functions $\varphi_l(s)$. Therefore, repeating proof of Theorem from [3], we find that the probability measure

$$\mu_N\big(W(s+ikh)\in A\big),\quad A\in\mathcal{B}(H(\hat{D})), \tag{11}$$

converges weakly to the distribution of the random element

$$W(s,\omega) = \sum_{k=1}^{\infty} \frac{a_k \omega(k)}{k^s}.$$

Here $\omega(k)$ is defined by the formula

$$\omega(k) = \prod_{p^\alpha \| k} \omega^\alpha(p),$$

where $p^\alpha \| k$ means that $p^\alpha | k$ but $p^{\alpha+1} \not| k$. If $s \in D$, we have

$$\varphi_l(s,\omega) = \sum_{k=1}^{\infty} \frac{c_l(k)\omega(k)}{k^s}, \quad l = 1,\ldots,r.$$

Then the definition of u, in view (10), yield

$$\begin{aligned} W(s,\omega) &= \sum_{l=1}^{r} \sum_{m=1}^{n} u_{lm} \sum_{k=1}^{\infty} \frac{c_l(k)\omega(k)}{k^{s_m+s}} \\ &= \sum_{l=1}^{r} \sum_{m=1}^{n} u_{lm} \varphi_l(s_m + s, \omega) = u\big(\Phi(s,\omega)\big). \end{aligned}$$

Therefore, the measure (11) converges weakly to the distribution of the random element $u\big(\Phi(s,\omega)\big)$ as $N \to \infty$. Hence we have the assertion of the lemma.

Proof of Theorem. By Lemma 2 we have

$$W(s + i\theta_{N_1}) \xrightarrow[N_1 \to \infty]{\mathcal{D}} u(\Phi),$$

where N_1 is the same as in proof Lemma 2. From this and (9) we have that

$$u(\Phi) \overset{\mathcal{D}}{=} u(\Phi_1). \tag{12}$$

Now let $u_1 : H(\hat{D}) \to \mathbb{C}$ be defined by the formula

$$u_1(f) = f(0), \quad f \in H(\hat{D}).$$

Then the function u_1 is measurable. Consequently, in virtue of (12)

$$u(\Phi)(0) \overset{\mathcal{D}}{=} u(\Phi_1)(0).$$

Then by the definition of u we find

$$\sum_{l=1}^{r} \sum_{m=1}^{n} u_{lm} \varphi_l(s_m) \overset{\mathcal{D}}{=} \sum_{l=1}^{r} \sum_{m=1}^{n} u_{lm} \varphi_{1l}(s_m) \tag{13}$$

for arbitrary complex numbers u_{lm}. A hyperplane in $\mathbb{R}^{2rn}$ form a determining class [1]. Then, the hyperplanes also form a determining class in the space $\mathbb{C}^{rn}$. Therefore, in view of (13), we see that $\mathbb{C}^{rn}$-valued random elements $\varphi_l(s_m)$ and $\varphi_{1l}(s_m)$, $l = 1,\dots,r$, $m = 1,\dots,n$, have the same distribution.

Let K be a compact subset of D, and let $f_1,\dots,f_r \in H(D)$. For any $\varepsilon > 0$

$$G = \Big\{(u_1,\dots,u_r) \in H^r(D) : \sup_{s\in K} \big|u_l(s) - f_l(s)\big| \leqslant \varepsilon, \quad l = 1,\dots,r\Big\}.$$

Let $\{s_m\}$ be a sequence dense in K. Moreover, let

$$\begin{aligned} G_n = \{&(u_1,\dots,u_r) \in H^r(D) : |u_l(s_m) - f_l(s_m)| \leqslant \varepsilon,\\ &l = 1,\dots,r, \quad m = 1,\dots,n\}. \end{aligned}$$

Then the properties of the random elements $\varphi_l(s_m)$ and $\varphi_{1l}(s_m)$ yield

$$m_H\big(\omega \in \Omega:\ \Phi(s,\omega) \in G_n\big) = P\big(\Phi_1(s) \in G_n\big). \tag{14}$$

Since the sequence $\{s_m\}$ is dense in K, we have $G_n \to G$ as $n \to \infty$. Therefore, if $n \to \infty$ in (14), we obtain

$$m_H\big(\omega \in \Omega:\ \Phi(s,\omega) \in G\big) = P\big(\Phi_1(s) \in G\big). \tag{15}$$

The space $H^r(D)$ is separable. Thus finite intersections of the spheres form a determining class [1]. Hence and from (14) we obtain

$$\Phi \overset{\mathcal{D}}{=} \Phi_1.$$

From this and (8) we have

$$\Phi_{N_1} \xrightarrow[N_1\to\infty]{\mathcal{D}} \Phi. \tag{16}$$

Therefore, the measure P_{N_1} converges weakly to the distribution of random element Φ as $N_1 \to \infty$. Consequently, the assertion of Theorem we obtain from Lemma 1 and Theorem 1.1.9 [5], since the random element Φ in (16) is independent on the choice of the sequence N_1.

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Daugiamatė diskrečioji ribinė teorema Matsumoto dzeta funkcijai analizinių funkcijų erdvėje

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Straipsnyje įrodoma daugiamatė diskrečioji ribinė teorema Matsumoto dzeta funkcijai tikimybinių matų silpno konvergavimo prasme analizinių funkcijų erdvėje.