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On the mean square for the periodic zeta–function on the critical line

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Let $s = \sigma + it$ be a complex variable, and let $\mathfrak{A} = \{a_m,\ m \in \mathcal{Z}\}$ be a sequence of complex numbers with period $k > 0$. The periodic zeta-function $\zeta(s;\mathfrak{A})$, for $\sigma > 1$, is defined by

$$\zeta(s;\mathfrak{A}) = \sum_{m=1}^{\infty} \frac{a_m}{m^s},$$

and by analytic continuation elsewhere. It was introduced and studied by B.C. Berndt and L. Schoenfeld in [1]. The papers [2] and [3] are devoted to asymptotics of the mean square

$$\int_0^T \left|\zeta(\sigma + it;\mathfrak{A})\right|^2 \mathrm{d}t, \quad T \to \infty,$$

in the critical strip. For this aim in [2] an approximation of $\zeta(s;\mathfrak{A})$ by a finite sum was used, while in [3] an approximate functional equation for $\zeta(s;\mathfrak{A})$ was applied. An asymptotic formula of [3] is more precise than in [2]. The aim of this note is to apply an approximate functional equation in the case $\sigma = \frac{1}{2}$.

Let

$$K(k) = \sum_{q=1}^{k} |a_q|^2;$$

$$K_1(k) = \sum_{q=1}^{k} q|a_q|^2 \sum_{m=1}^{\infty} \frac{1}{m(mk+q)};$$

$$K_2(k) = k\sum_{q=1}^{k} \frac{|a_q|^2}{q}.$$

Theorem. *Let $T \to \infty$. Then*

$$\int_0^T \left|\zeta(\tfrac{1}{2} + it;\mathfrak{A})\right|^2 \mathrm{d}t = k^{-1}K(k)T\log T + k^{-1}K(k)T(2\gamma - \log\pi - 1)$$

$$-k^{-1}T(K_1(k)-K_2(k))+Bk^{\frac{1}{2}}K(k)T^{\frac{1}{2}}\log T+BkK(k).$$

Proof. Using the notation of [3], we have Using the notation of [3], we have

$$\int_{\frac{T}{2}}^{T}\left|\zeta\left(\frac{1}{2}+it;\mathfrak{A}\right)\right|^2 \mathrm{d}t = k^{-1}\int_{\frac{T}{2}}^{T}\left|\sum_{q=1}^{k}a_q\sum_{0\leqslant m\leqslant r}\frac{1}{(m+q/k)^{1/2+it}}\right|^2\mathrm{d}t$$

$$+k^{-1}\int_{\frac{T}{2}}^{T}\left|\sum_{q=1}^{k}a_q\sum_{1\leqslant m\leqslant n}\mathrm{e}^{-2\pi imq/k}\frac{1}{m^{1/2-it}}\right|^2\mathrm{d}t$$

$$+Bk^{-1}\left|\sum_{q=1}^{k}a_q\right|^2\int_{\frac{T}{2}}^{T}t^{-1/2}\mathrm{d}t+BK(k)\int_{\frac{T}{2}}^{T}t^{-3/2}\mathrm{d}t$$

$$+Bk^{-1}\left|\int_{\frac{T}{2}}^{T}\left(\frac{2\pi}{t}\right)^{it}\mathrm{e}^{-it}\sum_{q_1=1}^{k}a_{q_1}\sum_{q_2=1}^{k}\overline{a}_{q_2}\right.$$

$$\left.\times\sum_{0\leqslant m_1\leqslant r}\frac{1}{(m_1+\frac{q_1}{k})^{1/2+it}}\sum_{1\leqslant m_2\leqslant n}\mathrm{e}^{2\pi imq_2/k}\frac{1}{m_2^{1/2+it}}\mathrm{d}t\right|$$

$$+Bk^{-1}\left|\int_{\frac{T}{2}}^{T}\left(\frac{2\pi}{t}\right)^{1/4}\sum_{q_1=1}^{k}\overline{a}_{q_1}\sum_{q_2=1}^{k}a_{q_2}\sum_{0\leqslant m\leqslant r}\frac{1}{(m+\frac{q_1}{k})^{1/2-it}}\right.$$

$$\left.\times\exp\left\{if\left(\frac{q_2}{k},t\right)\right\}\psi\left(2y-2n+l-\frac{q_2}{k}\right)\mathrm{d}t\right|$$

$$+Bk^{-1}\left|\int_{\frac{T}{2}}^{T}\left(\frac{2\pi}{t}\right)^{1/4-it}\mathrm{e}^{-it}\sum_{q_1=1}^{k}\overline{a}_{q_1}\sum_{q_2=1}^{k}a_{q_2}\sum_{1\leqslant m\leqslant n}\mathrm{e}^{2\pi imq_1/k}\frac{1}{m^{1/2+it}}\right.$$

$$\left.\times\exp\left\{if\left(\frac{q_2}{k},t\right)\right\}\psi\left(2y-2n+l-\frac{q_2}{k}\right)\mathrm{d}t\right|$$

$$+Bk^{-1}\left|\int_{\frac{T}{2}}^{T}\sum_{q=1}^{k}a_q\sum_{0\leqslant m\leqslant r}\frac{1}{(m+\frac{q}{k})^{1/2+it}}\overline{R}(s,k)\mathrm{d}t\right|$$

$$+Bk^{-1}\left|\int_{\frac{T}{2}}^{T}\left(\frac{2\pi}{t}\right)^{it}\sum_{q=1}^{k}a_q\sum_{1\leqslant m\leqslant n}\mathrm{e}^{-2\pi imq/k}\frac{1}{m^{1/2-it}}\overline{R}(s,k)\mathrm{d}t\right|$$

$$+Bk^{-1}\Bigg|\int_{\frac{T}{2}}^{T}\left(\frac{2\pi}{t}\right)^{1/4}\sum_{q=1}^{k}a_q\mathrm{e}^{if(\frac{q}{k},t)}\psi\Big(2y-2n+l-\frac{q}{k}\Big)\overline{R}(s,k)\mathrm{d}t\Bigg| \overset{def}{=} \sum_{j=1}^{10}I_j. \quad (1)$$

Let $T_1 = \max\big(\frac{T}{2}, 2\pi(m_1+\frac{q_1}{k})^2, 2\pi(m_2+\frac{q_2}{k})^2\big)$. Then, taking $M_q = \big[\big(\frac{T}{2\pi}\big)^{1/2}-\frac{q}{k}\big]$, we find

$$\begin{aligned}I_1 &= k^{-1}\sum_{q=1}^{k}|a_q|^2\sum_{0\leqslant m\leqslant M_q}\int_{T_1}^{T}\frac{\mathrm{d}t}{m+\frac{q}{k}}\\
&+k^{-1}\sum_{q=1}^{k}|a_q|^2\sum_{0\leqslant m_1\leqslant M_q}\sum_{0\leqslant m_2\leqslant M_q}\frac{1}{(m_1+\frac{q_1}{k})^{1/2}(m_2+\frac{q_2}{k})^{1/2}}\\
&\times\int_{T_1}^{T}\left(\frac{m_1+\frac{q}{k}}{m_2+\frac{q}{k}}\right)^{it}\mathrm{d}t+k^{-1}\sum_{q_1=1}^{k}a_{q_1}\sum_{q_2=1}^{k}\overline{a}_{q_2}\\
&\times\sum_{\substack{0\leqslant m_1\leqslant M_{q_1}\\ m_1\neq m_2}}\sum_{0\leqslant m_2\leqslant M_{q_2}}\frac{1}{(m_1+\frac{q_1}{k})^{1/2}(m_2+\frac{q_2}{k})^{1/2}}\int_{T_1}^{T}\left(\frac{m_2+\frac{q_2}{k}}{m_1+\frac{q_1}{k}}\right)^{it}\mathrm{d}t\\
&+k^{-1}\sum_{\substack{q_1=1\\ q_1\neq q_2}}^{k}a_{q_1}\sum_{q_2=1}^{k}\overline{a}_{q_2}\sum_{0\leqslant m\leqslant\max(M_{q_1},M_{q_2})}\frac{1}{(m+\frac{q_1}{k})^{1/2}(m+\frac{q_2}{k})^{1/2}}\\
&\times\int_{T_1}^{T}\left(\frac{m+\frac{q_2}{k}}{m+\frac{q_1}{k}}\right)^{it}\mathrm{d}t \overset{def}{=} \sum_{j=1}^{4}I_{1j}. \end{aligned} \quad (2)$$

It is not difficult to see that for sufficiently large T

$$\begin{aligned}I_{11} &= k^{-1}\sum_{q=1}^{k}|a_q|^2\sum_{0\leqslant m\leqslant M_q}\frac{1}{m+\frac{q}{k}}\left(T-\max\left(\frac{T}{2},2\pi\Big(m+\frac{q}{k}\Big)^2\right)\right)\\
&= k^{-1}T\sum_{q=1}^{k}|a_q|^2\sum_{0\leqslant m\leqslant M_q}\frac{1}{m+\frac{q}{k}}-\frac{T}{2}\sum_{q=1}^{k}\frac{|a_q|^2}{q}\\
&-\frac{T}{2k}\sum_{q=1}^{k}|a_q|^2\sum_{1\leqslant m\leqslant\frac{\sqrt{T}}{2\sqrt{\pi}}-\frac{q}{k}}\frac{1}{m+\frac{q}{k}}-2\pi k^{-1}\sum_{q=1}^{k}|a_q|^2\sum_{\substack{1\leqslant m\leqslant M_q\\ m>\frac{\sqrt{T}}{2\sqrt{\pi}}-\frac{q}{k}}}\Big(m+\frac{q}{k}\Big). \end{aligned} \quad (3)$$

By the formula

$$\sum_{m\leqslant x}\frac{1}{m}=\log x+\gamma+\frac{B}{x},$$

where γ is the Euler constant, we have

$$\sum_{0\leqslant m\leqslant M_q} \frac{1}{m+\frac{q}{k}} = \frac{k}{q} + \sum_{1\leqslant m\leqslant M_q} \frac{1}{m+\frac{q}{k}} = \frac{k}{q} + \sum_{1\leqslant m\leqslant M_q} \frac{1}{m} - q \sum_{1\leqslant m\leqslant M_q} \frac{1}{m(mk+q)}$$
$$= \frac{k}{q} + \frac{1}{2}\log T - \log\sqrt{2\pi} + \gamma - q\sum_{m=1}^{\infty} \frac{1}{m(mk+q)} + \frac{B}{\sqrt{T}}.$$

Similarly

$$\sum_{1\leqslant m\leqslant \frac{\sqrt{T}}{2\sqrt{\pi}}-\frac{q}{k}} \frac{1}{m+\frac{q}{k}} = \frac{1}{2}\log T - \log\sqrt{4\pi} + \gamma - q\sum_{m=1}^{\infty} \frac{1}{m(mk+q)} + \frac{B}{\sqrt{T}}.$$

Moreover,

$$\sum_{\substack{1\leqslant m\leqslant M_q \\ m>\frac{\sqrt{T}}{2\sqrt{\pi}}-\frac{q}{k}}} \left(m+\frac{q}{k}\right) = \frac{\left[\sqrt{\frac{T}{2\pi}}-\frac{q}{k}\right]\left(\left[\sqrt{\frac{T}{2\pi}}-\frac{q}{k}\right]+1\right)}{2}$$
$$-\frac{\left[\sqrt{\frac{T}{4\pi}}-\frac{q}{k}\right]\left(\left[\sqrt{\frac{T}{4\pi}}-\frac{q}{k}\right]+1\right)}{2}$$
$$+B\sqrt{T} = \frac{T}{4\pi} - \frac{T}{8\pi} + B\sqrt{T} = \frac{T}{8\pi} + B\sqrt{T}.$$

All these estimates together with (3) yield

$$I_{11} = \frac{K(k)}{4k}T\log T + \frac{T}{2k}\left(K(k)\left(\gamma - \log\sqrt{\pi} - \frac{1}{2}\right)\right.$$
$$\left. - K_1(k) + K_2(k)\right) + Bk^{-1}K(k)T^{1/2}. \tag{4}$$

For the integrals $I_{1j}, j \geqslant 2$, we can apply the estimates of [3] with $\sigma = \frac{1}{2}$. This gives

$$I_{12} = Bk^{-1/2}K(k)T^{1/4} + Bk^{-1}K(k)T^{1/2}\log T, \tag{5}$$
$$I_{13} = Bk^{1/2}K(k)T^{1/4} + BK(k)T^{1/2}\log T, \tag{6}$$
$$I_{14} = BkK(k). \tag{7}$$

Therefore (2) and (4)–(7) show that

$$I_1 = \frac{K(k)}{4k}T\log T + \frac{T}{2k}\left(K(k)\left(\gamma - \log\sqrt{\pi} - \frac{1}{2}\right)\right.$$
$$\left. - K_1(k)+K_2(k)\right) + BK(k)T^{1/2}\log T + Bk^{1/2}K(k)T^{1/4} + BkK(k). \tag{8}$$

Now let $T_2 = \max\left(\frac{T}{2}, 2\pi m_1^2, 2\pi m_2^2\right)$. Then

$$\begin{aligned}
I_2 &= k^{-1}\sum_{q=1}^{k}|a_q|^2 \sum_{1\leqslant m\leqslant M_0} \frac{1}{m}\left(T - \max\left(\frac{T}{2}, 2\pi m^2\right)\right) \\
&\quad + Bk^{-1}\sum_{q=1}^{k}|a_q|^2 \sum_{1\leqslant m_1\leqslant m_2\leqslant M_0} \frac{m_2}{(m_1 m_2)^{1/2}(m_2 - m_1)} \\
&\quad + Bk^{-1}\sum_{\substack{q_1=1\\ q_1\neq q_2}}^{k}\sum_{q_2=1}^{k}|a_{q_1}\bar{a}_{q_2}| \sum_{1\leqslant m_1\leqslant m_2\leqslant M_0} \frac{m_2}{(m_1 m_2)^{1/2}(m_2 - m_1)} \\
&\quad + Bk^{-1}\sum_{\substack{q_1=1\\ q_1\neq q_2}}^{k}\sum_{q_2=1}^{k}|a_{q_1}\bar{a}_{q_2}| \sum_{1\leqslant m\leqslant M_0} \frac{1}{m}\sum_{j=1}^{4} I_{2j}. \qquad (9)
\end{aligned}$$

Similarly to the case of I_{11} we find

$$\begin{aligned}
I_{21} &= k^{-1}K(k)T\left(\log M_0 + \gamma + \frac{B}{\sqrt{T}}\right) - k^{-1}K(k)\frac{T}{2}\sum_{1\leqslant m\leqslant\sqrt{\frac{T}{4\pi}}}\frac{1}{m} \\
&\quad - 2\pi k^{-1}K(k)\sum_{\substack{1\leqslant m\leqslant M_0\\ m>\sqrt{\frac{T}{4\pi}}}} m = \frac{K(k)}{4k}T\log T \\
&\quad + \frac{T}{2k}K(k)\left(\gamma - \log\sqrt{\pi} - \frac{1}{2}\right) + Bk^{-1}K(k)T^{1/2}. \qquad (10)
\end{aligned}$$

Moreover,

$$\begin{aligned}
I_{22} &= Bk^{-1}K(k)T^{1/2}\log T, \\
I_{23} &= BK(k)T^{1/2}\log T, \\
I_{24} &= BK(k)\log T.
\end{aligned}$$

This and (9), (10) give

$$I_2 = \frac{K(k)}{4k}T\log T + \frac{T}{2k}K(k)\left(\gamma - \log\sqrt{\pi} - \frac{1}{2}\right) + BK(k)T^{1/2}\log T. \qquad (11)$$

The integrals I_j, $j \geqslant 3$, are estimated by the same way like in [3]. We have

$$\begin{aligned}
I_3 &= BK(k)T^{1/2}, \\
I_4 &= BK(k)T^{-1/2}, \\
I_5 &= Bk^{1/2}K(k)T^{1/2} + BK(k)T^{1/2}\log T, \\
I_6 &= Bk^{1/2}K(k)T^{1/4}\log T + BK(k)T^{1/2}\log T,
\end{aligned}$$

$$I_7 = BK(k)T^{1/2}\log T,$$
$$I_8 + I_9 + I_{10} = Bk^{1/2}K(k)T^{1/2}\log T.$$

Now hence and from (1), (8) and (11) we deduce that

$$\int_{\frac{T}{2}}^{T} \left|\zeta(\frac{1}{2} + it; \mathfrak{A})\right|^2 dt = \frac{1}{2k}K(k)T\log T + \frac{T}{k}K(k)\left(\gamma - \log\sqrt{\pi} - \frac{1}{2}\right)$$
$$-\frac{T}{2k}\big(K_1(k) - K_2(k)\big) + Bk^{1/2}K(k)T^{1/2}\log T + BkK(k).$$

Taking in the later formula $2^{-\alpha}T$ instead of T and summing over all non-negative integers α, we obtain the assertion of the theorem.

References

[1] B.C. Berndt, L. Schoenfeld, Periodic analogues of the Euler-Maclaurin and Poisson summation formulas with applications to number theory, *Acta Arith.*, **28**, 23–68 (1975).
[2] A. Kačėnas, A. Laurinčikas, On the periodic zeta-function, *Liet. Matem. Rink.* (to appear).
[3] A. Laurinčikas, D. Šiaučiūnas, On the periodic zeta-function. II, *Liet. Matem. Rink.* (to appear).

Periodinės dzeta funkcijos antrojo momento kritinėje tiesėje asimptotika

D. Šiaučiūnas

Straipsnyje gauta periodinės dzeta funkcijos antrojo momento asimptotika kritinėje tiesėje.