Liet. matem. rink., T. 41, spec. nr., 408–412

Some decidable classes of formulas of modal logic S4

Stanislovas NORGĖLA (VU)
e-mail: stasys.norgela@maf.vu.lt

1. Introduction

We will consider the formulas of modal logic without functional symbols. The formulas F contain only logical connectives $\neg$, $\wedge$, $\vee$ and no logical or modal symbol in F occur in the scope of a negation.

Consider a Gentzen-type formulation of the predicate calculus S4 from [1]. Its derivable objects are sequents $\Gamma \vdash$, where Γ is a finite list of formulas of the language considered. The order in Γ will always be disregarded, hence Γ is treated as a multiset (the number of occurences of formulas is important). The axioms are the sequents of the forms $\Gamma, A, \neg A \vdash$, where A is a formula. We will denote the bound individual variables by $x, y, z, x_1, x_2, \ldots$, free individual variable by $a, a_1, a_2, \ldots$, and terms by $t, t_1, t_2, \ldots$.

2. Basic results

DEFINITION 1. A calculus S4, in which the ground term t of the inference rule $(\forall \vdash)$

$$\frac{\Gamma, F(t), \forall x F(x) \vdash}{\Gamma, \forall x F(x) \vdash}$$

belongs to the sequent written under the line, is called minus-normal (if the conclusion does not contain any ground term, then t is a new free variable).

DEFINITION 2. Given a formula F(x), a substitution t/x, the formula F(t) is called a substitution instance of F and t an apparent term.

Theorem 1. *If a sequent is derivable in the calculus S4, then it is also derivable in the minus-normal calculus S4.*

Proof. Recall that we consider formulas without functional symbols. The theorem is not valid for the formulas with functional symbols. For example, the sequent $\forall x \forall y \Diamond \forall z \Diamond (P(x) \wedge (Q(y) \wedge \neg Q(f(z)))) \vdash$ is derivable in S4, but it is not derivable in the minus-normal calculus S4.

We can construct a derivation in S4 so that each application of the rule ($\exists\vdash$) the following condition hold: the apparently free variable is new not only in the considered branch, but also in all deduction tree. Let us fix the last sequent (that is an axiom) of a derivation tree. We begin a top-down path with the considered axiom. Assume, we found the first application of the rule ($\forall\vdash$) which does not satisfy the minus-normal condition. It means that free variables belong to the conclusion of the rule (suppose that one of these variables is a), but the apparent term t is a new variable not occurring in the conclusion. In this case, we replace t by a in the premises of the considered rule, that is, in each sequent over the line. After this replacing, the tree rests a derivation. Indeed,

1. The apparent variables in any application of the inference rule ($\exists\vdash$) are new variables in the deduction tree. From this it follows that all the considered variables differ from a.

2. Each application of the inference rule ($\forall\vdash$), which is over the considered application in the tree, is minus-normal. If an apparent term of some rule ($\forall\vdash$) were t, then now it would become a.

Note that the complexity of a derivation tree can only decrease after the renaming t by a. It is also possible to eliminate all applications of the inference rule ($\forall\vdash$) not satisfying the minus-normal condition.

DEFINITION 3. A formula of the form $R_1...R_nG$, where R_i $(i = 1, ..., n)$ is a quantifier or a modal operator, and G a quantifier-free formula not containing modal operators, is called a formula in the prenex form with a generalized prefix.

We use those notations throughout this paper.

DEFINITION 4. Given a formula $F = R_1...R_nG$ in prenex form with generalized prefix, the formula $R_{i_1}...R_{i_s}G$, where $R_{i_1}...R_{i_s}$ is obtained from $R_1...R_n$ by deleting all occurrences of modal operators, is called a projection of F.

Theorem 2. *Let there be a formula $R_1...R_nG$ in the prenex form with a generalized prefix such that the prefix does not contain modal operator $\Diamond$. Then $R_1...R_nG\vdash$ is derivable if and only if its projection $R_{i_1}...R_{i_s}G\vdash$ is derivable.*

Proof. Application of the rule ($\Box\vdash$) gives us only a possibility to repeat the applications of the rules ($\forall\vdash$), ($\exists\vdash$) with the same main formula $R_1...R_nG$ $(i \leqslant n)$ or $G\sigma$, where σ is a substitution. One does not need a special duplication rule because applications of the rule ($\forall\vdash$) preserve the main formulas in premises. Duplication of the formulas beginning with $\forall$ and also the formulas $G\sigma$ is not necessary because the weakening rule in a classical predicate calculus is superfluous. The theorem is proved.

Consider some decidable class A of formulas of a classical predicate calculus in prenex form. We will construct another class A_M of formulas in prenex form with a generalized prefix of modal logic using the set A.

DEFINITION 5. Let the modal operator $\Diamond$ does not occur in generalized prefix $R_1...R_n$ or there are occurrences of operator $\Diamond$ rightwards of the last occurrences of operator $\Box$. Formula $R_1...R_nG$ belongs to A_M if and only if its projection $R_{i_1}...R_{i_s}G$ belong to the class A.

Theorem 3. *The class A_M is decidable.*

Proof. Case 1. The modal operator $\Diamond$ does not occur in a generalized prefix of the formula. In this case, $R_1...R_nG$ is derivable if and only if $R_{i_1}...R_{i_s}G$ is derivable according to Theorem 2.

Case 2. The modal operator $\Box$ does not occur in a generalized prefix of a formula or there are occurrences of operator $\Diamond$ rightwards of the last occurrence of operator $\Box$, i.e., we consider the formulas having the shape of

$$R_1...R_i\Diamond\forall x_{i+2}...\forall x_{i+s-1}\exists x_s Q_{s+1}x_{s+1}...Q_nx_n.G \qquad (1)$$

The operator $\Diamond$ is in the $(i+1)$th position and there are only quantifiers rightwards. Assume that the first right occurrence of the quantifier $\exists$ with respect to the last occurrence of operator $\Diamond$ is in the $(i+s)$th position. Moreover, assume we have a matrix G in a conjunctive normal form with k clauses. Without loss of generality, we consider a calculus with the following propositional rule:

$$(\vee\wedge\vdash) \qquad \frac{\Gamma, L_1^1, ..., L_{n_1}^1 \vdash ... \Gamma, L_1^k, ..., L_{n_m}^k \vdash}{\Gamma, (L_1^1 \wedge ... \wedge L_{n_1}^1) \vee ... \vee (L_1^k \wedge ... \wedge L_{n_m}^k) \vdash}$$

where L_j^i $(i = 1, ..., k; j = 1, 2, ..., \max(n_1, ..., n_m))$ is a literal.

We will use the following proof-search strategy: whenever $G\sigma$ is obtained, we apply the rule $(\vee\wedge\vdash)$. Since the formula $\Gamma\sigma$ does not contain a modal operators, the strategy is complete. Let formula (1) be deducible and we have its deduction tree. Consider the k axioms obtained as a result of the application of the rule $(\vee\wedge\vdash)$. We follow the path from those axioms to the root. Suppose we found the first application of the rule $(\Diamond\vdash)$. We delete all atomic formulas by applying the rule $(\Diamond\vdash)$ and, therefore, each complementary pair can be only a successor of those applications of the rules, whose main formulas are of the form $R_v...R_nG$ $(v \geqslant i+2)$. That is, formula (1) is deducible if and only if there is such a substitution

$$\sigma = \{t_1/x_1, ..., t_i/x_i, t_{i+2}/x_{i+2}, ..., t_{i+s-1}/x_{i+s-1}\},$$

for which the formula $\exists x_{i+s}R_{i+s+1}...R_nG\sigma$ is deducible.

From Theorem 1 it follows that the set of the considered substitutions is finite up to renaming variables. Therefore, a class A_M is decidable because a deducibility of formulas of A_M reduces to a deducibility of formulas of the decidable class A. Theorem is proved.

Theorem 4. *Consider a class of formulas in the prenex normal form. If the matrix of formulas contain only one-place predicate variables and do not contain the occurrences of the modal operator $\Box$, then the class is decidable.*

Proof. Note that only atomic formulas are in the scope of a negation (see the introduction of this paper).

Let us fix a formula $F = R_1 ... R_n M$, where M is a matrix, that is, a quantifier-free formula. We will now find such a formula G of the classical calculus that the sequent $F \vdash$ is deducible in the modal logic S4 if and only if the sequent $G \vdash$ is deducible in classical predicate calculus. We will say that a quantifier-free formula M is in an m-disjunctive normal form if it is of the form $N_1 \vee ... \vee N_s$, where N_i $(i = 1, ..., s)$ is a conjunction of literals and the formulas beginning with $\Diamond$.

Using the well-known equivalences of classical propositional logic and the following equivalences of modal logic S4

$$\Diamond\Diamond A \equiv \Diamond A$$

$$\Diamond(A \vee B) \equiv \Diamond A \vee \Diamond B$$

we can obtain an equivalent formula M' which is in an m-disjunctive normal form and for any of its subformulas $\Diamond A$, that is, for any subformulas beginning with $\Diamond$ the following condition hold: A is also in an m-disjunctive normal form.

Below M' is a formula in an m-disjunctive normal form equivalent to matrix M of the initial formula F.

In that follows "subformula" means "subformula with the fixed occurrence". In other words, we consider different occurrences of the same formula as different subformulas. We define the transformation *El*, that is, an elimination of the operator $\Diamond$ by induction on the subformulas of M.

Suppose that $\Diamond G_1, ..., \Diamond G_n$ is the list of the all subformulas of M the modal degree of which is equal to one and $v_{i_1}, ..., v_{i_n}$ are the new individual variables not occurring in M. We change the subformulas $\Diamond G_j (j = 1, ..., n)$ by the formulas whose are obtained from G_j by replacing the all one-place predicate variables by the two-place predicate variables with the same name. The variable v_{i_j} is the second individual variable for all subformulas of G_j.

The modal degree of the initial formula is greater than the modal degree of the obtained formula. If the obtained formula contains the modal operator, then we repeat the elimination procedure. Notice that the considered formula contains the two-place predicate variables, but we replace only one-place predicate variables by new individual variables not occurring in the considered formula.

We change the initial formula $F = R_1 ... R_n M'$ by $R_1 .. R_n \exists v_{i_1} ... \exists v_{i_s} El(M')$, where $v_{i_1}, ..., v_{i_s}$ are new variables introduced by the elimination of the operator $\Diamond$ in M'. We will show that the sequent $F \vdash$ is deducible in S4 if and only if the sequent

$$R_1 .. R_n \exists v_{i_1} ... \exists v_{i_s} El(M') \vdash \tag{2}$$

is deducible in the classical predicate calculus.

In fact, if we can find in a deduction tree of $F \vdash$ such a substitution σ that a sequent of the form $\Diamond G\sigma \vdash$ (hence, $\Diamond G$ is a subformula of F) is deducible, then we can also find such a substitution σ' that $El(G)\sigma' \vdash$ is deducible in the classical calculus and vice versa. It suffices to use the substitution $\sigma' = \sigma \cup \{a/v_{i_j}\}$, where a is a new free variable and v_{i_j} is a variable introduced by the elimination of the operator $\Diamond$. Note that a literal occurring in $\Diamond G\sigma$ can form a comlementary pair in a deduction tree only with the literals of the same $\Diamond G\sigma$. That is, the result depends only on σ. The formula $\Diamond G\sigma$ is equivalent to either the logical constant *false* or logical constant *true*.

It means that we obtain a formula of the classical predicate calculus with one- and two-place predicate variables, which is derivable if and only if the initial formula of modal logic is derivable in S4. The obtained formula of classical predicate calculus belongs to the derivable class K [2], because every F-prefix is of length 1 or ends with the quantifier $\exists$ (recall, we consider the formula on the left side of a sequent). The theorem is proved.

Concluding remarks

Theorem 4 can be generalized in the case, where the formula is not in a prenex normal form. Our requirement that the formulas contain only logical connectives $\neg, \vee, \wedge$, is not necessary. It is possible to consider a general cases. For this reason, one needs to introduce a notion of the negative (positive) occurrence of a subformula to a formula.

References

[1] R. Feys, *Modal Logics*, Paris (1965).
[2] S.Yu. Maslov, The inverse method of establiching deducibility for logical calculus, *Proceedings of the Steklov Institute of Mathematics*, **98**, 25–96 (1971).

Kai kurios išsprendžiamos modalumo logikos S4 klasės

S. Norgėla

Nagrinėjamos modalumo logikos S4 formulės, kuriose yra tik loginės operacijos $\neg, \vee, \wedge$ ir neiginys sutinkamas tik prieš atomines formules. Jei į tokias formules įeina vien tik modalumo operatoriai $\Box$ ir nėra jose $\Diamond$ (arba įeina vien tik $\Diamond$ ir nėra jose $\Box$), tai tokių formulių klasės išsprendžiamos. Be to, įrodyta, kad normaliosios priešdėlinės formos su vienviečiais predikatiniais kintamaisiais formulių klasė išsprendžiama, jei formulių matricose nėra modalumo operatoriaus $\Box$ įeičių.