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Uniform distribution on the four-dimensional torus. I

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1. Notation and results

Let $\Omega = \Omega_4$ be the four-dimensional torus

$$\mathbf{x} = (x_1, x_2, x_3, x_4), \quad 0 \leqslant x_i < 1,$$

with coordinate-wise summation modulo 1 and with Lebesgue measure μ. It is well known that the endomorphisms of the torus Ω are defined by the non-singular matrices V with integer elements by

$$T\mathbf{x} \equiv \mathbf{x}V (\operatorname{mod} 1).$$

An endomorphism T is ergodic if and only if its matrix V has no eigenvalues equal to one. Let $Q = [a_1, b_1] \times [a_2, b_2]$ be the rectangle of the Euclidean plane on which two functions $z = \varphi_1(x, y)$ and $w = \varphi_2(x, y)$ are defined. In this case the vector $(x, y, \varphi_1(x, y), \varphi_2(x, y))$ defines the surface Γ in $\mathbb{R}^4$. Let

$$K_i = \frac{\varphi''_{ix^2}\varphi''_{iy^2} - (\varphi''_{ixy})^2}{(1 + \varphi^2_{ix} + \varphi^2_{iy})^2}, \quad i = 1, 2, \tag{1}$$

i.e., K_i is the Gaussian (total) curvature of components of the surface Γ.

We suppose that the partial derivatives of the third order of functions $\varphi_i(x, y)$ exist when $(x, y) \in Q$.

Theorem. *Let the surface* $\Gamma = \{(x, y, z, w),\ z = \varphi_1(x, y),\ w = \varphi_2(x, y),\ (x, y) \in Q\}$ *have the nonzeroes curvatures* (1) *of components, and let the assumption* $K_1 \cdot K_2 \geqslant 0$ *holds. Moreover, the characteristic polynomial of the matrix* V *is irreducible over the field of rational numbers with different real roots. Then for almost all points* $(x, y) \in Q$ *with respect to the Lebesgue measure the set of vectors*

$$\{(x, y, \varphi_1(x, y), \varphi_2(x, y))V\}, \quad \{(x, y, \varphi_1(x, y), \varphi_2(x, y))V^2\}, \ldots$$

is uniformly distributed on the unit cube $[0, 1]^4$ *of the space* $\mathbb{R}^4$.

2. The main lemmas

We divide the proof of the theorem into lemmas, and in this part of the work we prove only five lemmas.

We need the class of functions introduced by Korobov in 1989. Let $\overline{\Omega}_s$ be the unit cube in s-dimensional Euclidean space $\mathbb{R}^s$. A continuous on $\overline{\Omega}_s$ function $f(\mathbf{x})$ belongs to the class $E_s^\alpha(c)$ if it's Fourier coefficients

$$a_{m_1\dots m_s} = \int_{\Omega_s} f(\mathbf{x})e^{-2\pi i\mathbf{m}\mathbf{x}}d\mathbf{x},$$

where $m_1, \dots, m_s$ are integers, satisfy the condition

$$|a_{m_1\dots m_s}| \leqslant \frac{c}{(\hat{m}_1 \dots \hat{m}_s)^\alpha}$$

with $\alpha > 1$, $c > 0$, and

$$\hat{m}_k = \begin{cases} 1 & \text{if } m_k = 0, \\ |m_k| & \text{if } m_k \neq 0,\ k = 1, \dots, s. \end{cases}$$

Lemma 1. *Let $\varepsilon_N^1, \dots, \varepsilon_N^s$ be real numbers, $N = 1, 2, \dots$; $\delta = \delta_N = \max_{1\leqslant i\leqslant N} |\varepsilon_N^i| \to 0$, $N \to \infty$, and $\alpha > 1$. Furthermore, $\varrho_1, \varrho_2, \dots, \varrho_\nu$ are algebraic numbers linearly independent over the field of rational numbers. Then for $f \in E_s^2(c)$ the following quadrature formula*

$$\frac{1}{N}\sum_{k=1}^{N} f(\{k(\varrho_1+\varepsilon_N^1)\}, \dots, \{k(\varrho_s+\varepsilon_N^s)\}) = \int_{\overline{\Omega}_s} f(\mathbf{x})d\mathbf{x} + O\Big(\frac{c}{N} + cN\delta\frac{\alpha-1}{1+\varepsilon}\Big), \quad (2)$$

holds, where $\varepsilon > 0$ is an arbitrary fixed number, the constant in the symbol "O" depends on α, ε, s, and on arithmetic properties of numbers $\varrho_1, \dots, \varrho_s$.

Proof. The assertion of the lemma is analoguous to that of Lemma 1 in [3], and follows from the following assertions.

Theorem A (Korobov, 1989). *Let $f(x_1, x_2, \dots, x_s) \in E_s^\alpha(c)$, $p > s$, is a prime number, and $(a_\nu, p) = 1$, $\nu = 1, 2, \dots, s$. Then*

$$\int_0^1 \dots \int_0^1 f(x_1 \dots x_s)\,dx_1 \dots dx_s = \frac{1}{p}\sum_{k=1}^{p} f\Big(\Big\{\frac{a_1 k}{p}\Big\}, \dots, \Big\{\frac{a_s k}{p}\Big\}\Big) + R_p(f),$$

where

$$|R_p(f)| \leqslant \sum_{m_1\dots m_s=-\infty}^{\infty}{}' \frac{\delta_p(a_1m_1 + \dots + a_sm_s)}{(\hat{m}_1 \dots \hat{m}_s)^\alpha}.$$

Here $\delta_q(a)$ is defined by

$$\delta_q(a) = \begin{cases} 1 & \text{if } a \equiv 0 \pmod q, \\ 0 & \text{if } a \not\equiv \pmod q. \end{cases}$$

The parallelepipedal net can be chosen in such manner that the inequality

$$|R_p(f) = O\left(\frac{\ln^{\alpha s} p}{p^\alpha}\right)$$

should be satisfied, the constant in O depends only on α and s.

Theorem B (Schmidt, 1983). *Let K be a real algebraic field. Then there exists a constant c_k such that for any real number $\alpha \notin K$ there exist infinitely many numbers $\beta \in K$ for which the inequality*

$$|\alpha - \beta| < c_k \max\left((1, |\alpha|^2) H_k(\beta)\right)^{-2}$$

holds. Here $H_K(\beta)$ is an absolute altitude of β in the field K.

Let us put

$$G_i(x, y) = \left(\frac{\partial \varphi_i}{\partial x} - a\right)^2 + \left(\frac{\partial \varphi_i}{\partial y} - b\right)^2.$$

Here a and b are real numbers.

Lemma 2. *Let (x_0, y_0) be a fixed point, and we put*

$$a_i = \frac{\partial \varphi_i}{\partial x}(x_0, y_0) \quad \text{and} \quad b_i = \frac{\partial \varphi_i}{\partial y}(x_0, y_0).$$

Let the total curvature K_i of the surface $z_i = \varphi_i(x, y)$, $i = 1, 2$, satisfy the condition $|K_i| \geqslant \chi_i > 0$. If $\max(|x - x_0|, |y - y_0|) \leqslant \delta_i$, then there exists a constant $c_i > 0$ such that

$$G_i(x, y) \geqslant c_i \min\left(|x - x_0|^2, |y - y_0|^2\right), \quad i = 1, 2.$$

Proof. The functions $\frac{\partial \varphi_i}{\partial x}$ and $\frac{\partial \varphi_i}{\partial y}$ have bounded derivatives of the second order. We take the Taylor series expansion of these functions

$$\begin{aligned} \frac{\partial \varphi_i}{\partial x} &= a + (x - x_0)\frac{\partial^2 \varphi_i}{\partial x^2}(x_0, y_0) + (y - y_0)\frac{\partial^2 \varphi_i}{\partial x \partial y}(x_0, y_0) \\ &\quad + B_{i1}\left(|x - x_0|^2 + |x - x_0||y - y_0| + |y - y_0|^2\right), \\ \frac{\partial \varphi_i}{\partial y} &= b + (y - y_0)\frac{\partial^2 \varphi_i}{\partial y^2}(x_0, y_0) + \frac{\partial^2 \varphi_i}{\partial x \partial y}(x_0, y_0)(x - x_0) \\ &\quad + B_{i2}\left(|x - x_0|^2 + |x - x_0||y - y_0| + |y - y_0|^2\right), \end{aligned} \tag{3}$$

where B_{i1} and B_{i2}, $i=1,2$, are bounded functions. From this we obtain

$$\begin{aligned} G_i &= \Big(\frac{\partial\varphi_i}{\partial x}-a_i\Big)^2+\Big(\frac{\partial\varphi_i}{\partial y}-b_i\Big)^2 \\ &= \Big(\Big(\frac{\partial^2\varphi_i}{\partial x^2}(x_{0i},y_{0i})\Big)^2+\Big(\frac{\partial^2\varphi_i}{\partial x\partial y}(x_{0i},y_{0i})\Big)^2\Big)(x-x_{0i})^2 \\ &\quad +2\frac{\partial^2\varphi_i}{\partial x\partial y}(x_{0i},y_{0i})\Big(\frac{\partial^2\varphi_i}{\partial x^2}(x_{0i},y_{0i})\frac{\partial^2\varphi_i}{\partial y^2}(x_{0i},y_{0i})\Big)(x-x_{0i})(y-y_{0i}) \\ &\quad +B_{i3}\big(|x-x_{0i}|^3+|x-x_{0i}|^2|y-y_{0i}|+|x-x_{0i}||y-y_{0i}|^2+|y-y_{0i}|^3\big), \end{aligned}$$

where B_{i3}, $i=1,2$ is bounded for $(x,y)\in D$. According to the conditions of the lemma and relation (3) we get

$$\begin{aligned} &-4\Big(\Big(\frac{\partial^2\varphi_i}{\partial x\partial y}(x_{0i},y_{0i})\Big)^2-\frac{\partial^2\varphi_i}{\partial x^2}(x_{0i},y_{0i})\frac{\partial^2\varphi_i}{\partial y^2}(x_{0i},y_{0i})\Big)^2 \\ &\quad = -4K_i^2(x_{0i},y_{0i})\Big(1+\Big(\frac{\partial\varphi_i(x_{0i},y_{0i})}{\partial x}\Big)^2+\Big(\frac{\partial\varphi_i(x_{0i},y_{0i})}{\partial y}\Big)^2\Big)^4 \\ &\quad \leqslant -4K^i(x_{0i},y_{0i})\leqslant -4\chi_i, \end{aligned}$$

and

$$G_i\geqslant \max\big(|x-x_{0i}|^2,|y-y_{0i}|^2\big)\big(c_{i4}\chi_i^3-B_{i4}|x-x_{0i}|+|y-y_{0i}|\big),$$

where $B_{i4}=3\max\limits_{x,y\in D}|B_{i3}|$ and $c_{i4}>0$, $i=1,2$.

The last assertion proves the lemma.

Lemma 3. *Let a_i and b_i be such that for $(x,y)\in Q$,*

$$G_i(x,y)=\Big(\frac{\partial\varphi_i}{\partial x}-a_i\Big)^2+\Big(\frac{\partial\varphi_i}{\partial y}-b_i\Big)^2\geqslant d_i^2>0 \tag{4}$$

where d_i, $i=1,2$, are fixed constants. Then there exists a quadratic net with the lines parallel to the coordinate axes, the sides of quadrates being of fixed lenght $c_{i5}d_i$, and for $(x,y)\in H_i\cap Q$ at least one of two assertions

$$\Big|\frac{\partial\varphi_i}{\partial x}-a_i\Big|\geqslant\frac{d_i}{2\sqrt2}\quad or\quad \Big|\frac{\partial\varphi_i}{\partial y}-b_i\Big|\geqslant\frac{d_i}{2\sqrt2}$$

holds.

Proof. Since $G_i(x,y)\geqslant d_i^2$, for an arbitrary point $(x_{1i},y_{1i})\in Q$,

$$\Big(\frac{\partial\varphi_i}{\partial x}-a\Big)^2\geqslant\frac{d_i}{2},\quad \Big|\frac{\partial\varphi_i}{\partial x}-a\Big|\geqslant\frac{d_i}{\sqrt2},$$

or

$$\left(\frac{\partial \varphi_i}{\partial y} - b\right)^2 \geqslant \frac{d_i^2}{2}, \quad \left|\frac{\partial \varphi_i}{\partial y} - b\right|^2 \geqslant \frac{d_i^2}{\sqrt{2}},$$

because in (4) at least one summand is more than a half of the right side. We examine the case

$$\left|\frac{\partial \varphi_i}{\partial x}(x_1, y_1) - a\right| \geqslant \frac{d_i^2}{\sqrt{2}}. \tag{5}$$

We take the Taylor series expansion

$$\frac{\partial \varphi_i}{\partial x} = \frac{\partial \varphi_i}{\partial x}(x_{i1}, y_{i1}) + \frac{\partial^2 \varphi_i}{\partial x^2}(\hat{x}_{i1}, \hat{y}_{i1})(x - x_1) + \frac{\partial^2 \varphi_i}{\partial x \partial y}(\hat{x}_{i1}, \hat{y}_{i1})(y - y_1),$$

which is true for some $\hat{x}_{i1}$, $\hat{y}_{i1}$. Therefore

$$\begin{aligned}\left|\frac{\partial \varphi_i}{\partial x}(x, y) - a\right| &\geqslant \left|\frac{\partial \varphi_i}{\partial x}(x_i, y_i) - a\right| \\ &\quad - \left|\frac{\partial^2 \varphi_i}{\partial x^2}(\hat{x}_{i1}, \hat{y}_{i1})(x - x_1) + \frac{\partial^2 \varphi_i}{\partial x \partial y}(\hat{x}_{i1}, \hat{y}_{i1})(y - y_1)\right| \\ &\geqslant \left|\frac{\partial \varphi_i}{\partial x}(x_i, y_i) - a\right| - B_{i5} \max\left(|x - x_1|, |y - y_1|\right),\end{aligned}$$

where

$$B_{i5} = \max_{x,y \in Q}\left(\left|\frac{\partial^2 \varphi_i}{\partial x^2}\right| + \left|\frac{\partial^2 \varphi_i}{\partial y^2}\right|\right) < \infty.$$

In view of (5), for $(x, y) \in Q$ and

$$\max\left(|x - x_{i1}|, |y - y_{i1}|\right) \leqslant \frac{d_i}{2B_{i5}\sqrt{2}} = c_{i7} d_i, \tag{6}$$

we get

$$\left|\frac{\partial \varphi_i}{\partial x} - a_i\right| \geqslant \frac{d_i}{\sqrt{2}}, \quad i = 1, 2.$$

Let the point (x_{i1}, y_{i1}) be the center of a square having common side with the square (6). Then similarly we get that, for $(x, y) \in D$,

$$\max\left(|x - x_{i2}|, |y - y_{i2}|\right) \leqslant c_{i7} d_i.$$

Thus we get the net satisfying the assertion of the lemma.

Lemma 4. *Let $\psi_1(t), \ldots, \psi_m(t)$ be m-times differentiable functions. Let $\mathbf{a} = (a_1, \ldots, a_m)$ denote a real nonzero vector, and in the interval $[a, b]$ the Wronskian $W[\psi_1, \ldots, \psi_m] > 0$. If the function $g(t) = a_1\psi_1 + \ldots + a_m\psi_m$ is equal to zero for $t = t_0$, then there exist λ_1 and λ_2 such that $|g(t)| \geqslant \lambda_1 |t - t_0|^{m-1}$ for $|t - t_0| \leqslant \lambda_2$, $\lambda_i > 0$, $i = 1, 2$.*

Proof. We put

$$M = \min_{a \leqslant t \leqslant b} \left| W[\psi_1, \ldots, \psi_m] \right|.$$

We consider the linear system

$$\sum_{i=1}^{m} a_i \psi_i^{(k)}(t) = g^{(k)}(t), \quad 0 \leqslant k \leqslant m - 1,$$

with respect to unknowns $a_1, \ldots, a_m$. When $t \in [a, b]$, the absolute value of the determinant of this system is not smaller than M, and the solution of the system is expressed by

$$a_i = \frac{1}{W[\psi_1, \ldots, \psi_m]} \begin{vmatrix} \psi_1 & \ldots & g(t) & \psi_m \\ \psi_1' & \ldots & g'(t) & \psi_m' \\ \ldots & \ldots & \ldots & \ldots \\ \psi_1^{(m-1)} & \ldots & g^{(m-1)}(t) & \psi_m^{(m-1)} \end{vmatrix}, \quad i = 1, \ldots, m.$$

Developing this determinant along the i-th column we get, for $t \in [a, b]$,

$$|a_i| \leqslant \frac{c}{M} \max_k \left| g^{(k)}(t) \right|.$$

Since $(a_1 \ldots a_m)$ is nonzero vector, we can find l, $0 \leqslant l \leqslant m - 1$, such that

$$\left| g^{(l)}(t_0) \right| \geqslant \frac{M}{c} \max_i |a_i| = \mu > 0. \tag{7}$$

Let us have the Taylor series expansion for $g(t)$

$$g(t) = \sum_{k=1}^{m-1} \frac{(t - t_0)^k}{k!} g^{(k)}(t_0) + \frac{(t - t_0)^m}{m!} g^{(m)}(\hat{t}), \tag{8}$$

where $\hat{t}$ is some point on $[a, b]$. Suppose that $g'(t_0) \neq 0$. Then by (8) we get

$$g(t) = (t - t_0)\left(g'(t_0) + (t - t_0)B(t)\right),$$

where $B(t)$ is bounded on $[a, b]$. Then, for $|t - t_0| \leqslant |g'(t_0)| \left(2 \max_t |B(t)|\right)^{-1}$, we have

$$|g(t)| \geqslant \frac{|g'(t_0)|}{2 \max_t |B(t)|} |t - t_0|. \tag{9}$$

If $g'(t_0) = \ldots = g^{(\nu)}(t_0) = 0$, $g^{(\nu+1)} \neq 0$, $\nu \leqslant m-2$, we find from (8), for $|t - t_0| \leqslant |g^{(\nu+1)}(t_0)|b^{-1}$, that

$$|g(t)| \geqslant \frac{|g^{(\nu+1)}(t_0)||t-t_0|^{\nu+1}}{B}. \tag{10}$$

The relations (9) and (10) complete the proof.

Corollary. *The function $g(t)$ has only finitely many zeros.*

Lemma 5. *Let θ be a real root of the characteristic polynomial of the matrix $V = \|a_{ik}\|$, and $\mathbf{w} = (w_1, \ldots, w_s)$ is the eigenvector corresponding to θ. If the polynomial is irreducible over the field of rational numbers, then the relation*

$$m_1 w_1 + \ldots + m_s w_s = 0$$

is posible if all m_i, $i = 1, \ldots, s$, are equal to zero.

The components of the vector $\mathbf{w}$ verify the homogenous system of equations with rank $s-1$:

$$\begin{aligned}
&(a_{11} - \theta)w_1 + a_{12}w_2 + \ldots + a_{1s}w_s = 0,\\
&a_{12}w_1 + (a_{12} - \theta)w_2 + \ldots + a_{2s}w_s = 0,\\
&\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots\\
&a_{s1}w_1 + a_{s2}w_2 + \ldots + (a_s s - \theta)w_s = 0.
\end{aligned} \tag{11}$$

Let we put $w_k = 1$ for some fixed k, $1 \leqslant k \leqslant s$. Thus we get the system of order $s-1$, and its determinant $p_k(\theta)$ has the same order $s-1$ and is not equal to zero. We can represent the solution of the last system in the form

$$w_1 = \frac{p_1(\theta)}{p_k(\theta)}, \ \ldots, \ w_{k-1} = \frac{p_{k-1}(\theta)}{p_k(\theta)}, \ w_{k+1} = \frac{p_{k+1}(\theta)}{p_k(\theta)}, \ \ldots, \ w_s = \frac{p_s(\theta)}{p_k(\theta)}.$$

Having in mind that $w_k = 1$, we see that the relation

$$m_1 w_1 + \ldots + m_s w_s = 0$$

is equivalent to

$$m_1 p_1(\theta) + \ldots + m_k p_k(\theta) + \ldots + m_s p_s(\theta) = 0.$$

Hence we get $m_k = 0$ by reason that $p_k(\theta)$ is of order $s-1$ when the different ones have the order $s-2$. Taking for k the values $1, 2, \ldots, s$, we get the assertion of lemma.

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Tolygūs pasiskirstymai ant keturmačio toro

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