LMD mokslo darbai, III tomas, 104–109

On the distributions of additive functions

Jonas ŠIAULYS * (VU)

1. Introduction

We shall consider the distribution functions $\nu_x\big(f_x(m) < u\big)$, where $\{f_x(m),\ x \geqslant 2\}$ is some set of strongly additive functions and

$$\nu_x\big(f_x(m) < u\big) = \frac{1}{[x]}\#\{m \leqslant x : f_x(m) < u\}.$$

In the papers [2–4] the weak convergence of these functions to the Poisson law was investigated. In this paper we examine the conditions for the weak convergence of $\nu_x\big(f_x(m) < u\big)$ to some distribution function $F(u)$. We investigate only those additive functions $f_x(m)$, for which $f_x(p) \in \{0, 1\}$ for primes p.

2. The main result

Theorem. *Let $\{f_x(m),\ x \geqslant 2\}$ be a set of strongly additive functions, $f_x(p) \in \{0,1\}$ for each prime number p. The distribution function $\nu_x\big(f_x(m) < u\big)$ converge weakly as $x \to \infty$ if and only if the finite limits*

$$\lim_{x\to\infty} \sum_{\substack{p_1 \leqslant x \\ f_x(p_1)=1}} \frac{1}{p_1} \cdots \sum_{\substack{p_{l-1} \leqslant x \\ p_{l-1} \neq p_1, p_2, \ldots, p_{l-2} \\ f_x(p_{l-1})=1}} \frac{1}{p_{l-1}} \sum_{\substack{p_l \leqslant x/p_1 p_2 \cdots p_{l-1} \\ p_l \neq p_1, p_2, \ldots, p_{l-1} \\ f_x(p_l)=1}} \frac{1}{p_l} = g_l$$

exist for each natural number l.

Moreover, in this case the limiting distribution has characteristic function

$$1 + \sum_{l=1}^{\infty} \frac{g_l}{l!}(e^{it} - 1)^l.$$

*Partially supported by Lithuanian State Studies and Science Foundation.

3. The proof of necessity

Let $F(u)$ is the limit distribution function for $\nu_x\big(f_x(m) < u\big)$. This function is a distribution function of some discrete random variable with jumps at non-negative integer numbers. Assume that $\varphi_k = F(k+0) - F(k)$. Then from the weak convergence we have

$$\lim_{x\to\infty} \nu_x\big(f_x(m) = k\big) = \varphi_k \tag{1}$$

for each non-negative integer k.

It is clear that $\varphi_{\hat{k}} > 0$ for some$\hat{k}$. Using the Halász's inequality (see [1])

$$\nu_x\big(h(m) = a\big) \leqslant c_1\Bigg(\sum_{\substack{p\leqslant x\\ h(p)\neq 0}} \frac{1}{p}\Bigg)^{-1/2},$$

which holds for every additive function $h(m)$ and every real number a, we obtain

$$\sum_{\substack{p\leqslant x\\ f_x(p)=1}} \frac{1}{p} \leqslant \frac{4c_1^2}{\varphi_{\hat{k}}^2}$$

for sufficiently large x.

Hence we get

$$\limsup_{x\to\infty} \sum_{p\leqslant x}{}^{*} \frac{1}{p} \leqslant c_2, \tag{2}$$

where c_2 may depend on the limit distribution function $F(u)$, and the sign $*$ means that the summation is taken over primes p for which $f_x(p) = 1$.

Let

$$\beta_x(l) = \frac{1}{x}\sum_{m\leqslant x} f_x(m)\big(f_x(m)-1\big)\ldots\big(f_x(m)-l+1\big)$$

for a natural number l.

An easy computation shows that

$$\begin{aligned}\beta_x(l) &= \frac{1}{x}\sum_{m\leqslant x}\ \sum_{p_1|m}{}^{*}\ \sum_{\substack{p_2|m\\ p_2\neq p_1}}{}^{*}\cdots \sum_{\substack{p_l|m\\ p_l\neq p_1,p_2,\ldots,p_{l-1}}}{}^{*} 1\\ &= \frac{1}{x}\sum_{p_1\leqslant x}{}^{*}\ \sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}{}^{*}\cdots \sum_{\substack{p_{l-1}\leqslant x\\ p_{l-1}\neq p_1,p_2,\ldots,p_{l-2}}}{}^{*}\ \sum_{\substack{p_l\leqslant x/p_1p_2\cdots p_{l-1}\\ p_l\neq p_1,p_2,\ldots,p_{l-1}}}{}^{*} \left[\frac{x}{p_lp_2\ldots p_l}\right].\end{aligned} \tag{3}$$

According to the inequality (2) we have

$$\limsup_{x\to\infty}\beta_x(l) \leqslant \limsup_{x\to\infty}\Big(\sum_{p\leqslant x}{}^{*}\frac{1}{p}\Big)^l \leqslant c_2^l \tag{4}$$

for each l.

Suppose now that l is a fixed natural number and $K \geqslant l+10$. We have

$$\begin{aligned}\beta_x(l) &= \frac{1}{x}\sum_{\substack{m\leqslant x\\ f_x(m)\leqslant K}} f_x(m)\big(f_x(m)-1\big)\dots\big(f_x(m)-l+1\big)\\ &+\frac{1}{x}\sum_{\substack{m\leqslant x\\ f_x(m)> K}} f_x(m)\big(f_x(m)-1\big)\dots\big(f_x(m)-l+1\big)\frac{f_x(m)-l}{f_x(m)-l}\\ &=\sum_{k=l}^{K} k(k-1)\dots(k-l+1)\frac{1}{x}\sum_{\substack{m\leqslant x\\ f_x(m)=k}}1+\frac{B}{K-l}\beta_x(l+1).\end{aligned}\tag{5}$$

It follows from (1) and (4) that

$$\limsup_{x\to\infty}\beta_x(l) = \limsup_{K\to\infty}\sum_{k=l}^{K} k(k-1)\dots(k-l+1)\varphi_k.$$

According to estimate (4) we see that sequence

$$g_{lK} = \sum_{k=l}^{K} k(k-1)\dots(k-l+1)\varphi_k$$

is increasing and bounded. Therefore the limit

$$g_l = \lim_{K\to\infty} g_{lK} = \sum_{k=l}^{\infty} k(k-1)\dots(k-l+1)\varphi_k$$

exist for each fixed natural l.

Hence from (4) and (5) we have

$$\lim_{x\to\infty}\beta_x(l) = g_l. \tag{6}$$

On the other hand (3) shows that

$$\begin{aligned}\beta_x(l) &= \sum_{p_1\leqslant x}{}^{*}\frac{1}{p_1}\sum_{\substack{p_2\leqslant x\\ p_2\neq p_1}}{}^{*}\frac{1}{p_2}\dots\sum_{\substack{p_{l-1}\leqslant x\\ p_{l-1}\neq p_1,p_2,\dots,p_{l-2}}}{}^{*}\frac{1}{p_{l-1}}\sum_{\substack{p_l\leqslant x/p_1p_2\cdots p_{l-1}\\ p_l\neq p_1,p_2,\dots,p_{l-1}}}{}^{*}\frac{1}{p_l}\\ &+B\,l!\,\nu_x\big(\omega(m)=l\big).\end{aligned}\tag{7}$$

Since

$$\nu_x(\omega(m) = l) \sim \frac{(\ln\ln x)^{l-1}}{(l-1)!\,\ln x}$$

for each fixed natural l (see, for example, [5] Ch.II), we conclude from (6) that the conditions of our theorem are satisfied.

4. The proof of sufficiency

Suppose now that the limit

$$g_l = \lim_{x\to\infty} \sum_{p_1 \leqslant x}^{*} \frac{1}{p_1} \sum_{\substack{p_2 \leqslant x \\ p_2 \neq p_1}}^{*} \frac{1}{p_2} \cdots \sum_{\substack{p_{l-1} \leqslant x \\ p_{l-1} \neq p_1, p_2, \ldots, p_{l-2}}}^{*} \frac{1}{p_{l-1}} \sum_{\substack{p_l \leqslant x/p_1 p_2 \cdots p_{l-1} \\ p_l \neq p_1, p_2, \ldots, p_{l-1}}}^{*} \frac{1}{p_l} \tag{8}$$

exist for each fixed natural l.

Applying (7) we can assert that

$$\lim_{x\to\infty} \hat{\beta}_x(l) = \lim_{x\to\infty} \beta_x(l) = g_l, \tag{9}$$

where l is a natural number and

$$\hat{\beta}_x(l) = \frac{1}{[x]} \sum_{m \leqslant x} f_x(m)\big(f_x(m) - 1\big) \ldots \big(f_x(m) - l + 1\big).$$

Let $\psi_x(t)$ is the characteristic function of $\nu_x\big(f_x(m) < u\big)$. It is clear that

$$\psi_x(t) = \frac{1}{[x]} \sum_{m \leqslant x} \mathrm{e}^{itf_x(m)}$$

for $x \geqslant 2$, $t \in \mathbb{R}$.

If r and n are natural numbers, then

$$\left|\mathrm{e}^{itr} - 1 - \sum_{j=1}^{n-1} \binom{r}{j} \left(\mathrm{e}^{it} - 1\right)^j\right| \leqslant \binom{r}{n} |\mathrm{e}^{it} - 1|^n.$$

Hence

$$\psi_x(t) = 1 + \sum_{l=1}^{L} \frac{(\mathrm{e}^{it} - 1)^l}{l!}\, \hat{\beta}_x(l) + \frac{B}{(L+1)!} |\mathrm{e}^{it} - 1|^{L+1} \hat{\beta}_x(L),$$

where $L \in \mathbb{N}$.

We obtain from (8) for the case $l = 1$

$$\lim_{x\to\infty} \sum_{p\leqslant x}^{*} \frac{1}{p} = g_1.$$

Repeated application of (8) enables us to write

$$g_l \leqslant \lim_{x\to\infty} \Big(\sum_{p\leqslant x}^{*} \frac{1}{p}\Big)^l = g_1^l$$

for each natural number l.

Therefore from (9) we have

$$\psi_x(t) = 1 + \sum_{l=1}^{L} \frac{(e^{it}-1)^l}{l!}\hat{\beta}_x(l) + \frac{B}{(L+1)!}|e^{it}-1|^{L+1} g_1^{L+1}, \tag{10}$$

where $t \in \mathbb{R}$, $x \geqslant 2$ and $L \in \mathbb{N}$.

Inequality $g_l \leqslant g_1^l$, $l \in \mathbb{N}$, shows that the series

$$\sum_{l=1}^{\infty} \frac{(e^{it}-1)^l}{l!} g_l$$

converges uniformly to some continuous function $\psi(t)$.

By (10) we obtain

$$\lim_{x\to\infty} \psi_x(t) = \psi(t)$$

for each real number t.

Since $\psi(t)$ is continuous, it follows from the last equality that distribution functions $\nu_x(f_x(m) < u)$ converge weakly to some distribution function $F(u)$, which has its characteristic function $\psi(t)$. This completes the prof.

5. Concluding remarks

It can be seen, that the limit distribution function $F(u)$ from our theorem has some special properties. We can obtain after some calculations the following statements.

1. $F(u)$ is distribution function of some discret random variable with jumps at non-negative numbers.
2. There exist the factorial moments

$$g_n = \int_{-\infty}^{+\infty} u(u-1)\dots(u-n+1)\mathrm{d}F(u)$$

for each natural n.

3. The factorial moments g_n satisfty the inequalities

$$g_n \leqslant g_{n-k} g_k, \quad k = 0, 1, \ldots, n-1.$$

4. If $g_l = 0$ for some $l \geqslant 2$, then $g_1 \leqslant \ln l$.

I am sure that $F(u)$ has more special properties. I think only a few of distribution functions can occur as weak limits in our theorem.

References

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Apie adityviųjų funkcijų skirstinius

J. Šiaulys

Darbe gautos būtinos ir pakankamos sąlygos skirstinių

$$\nu_x(f_x(m) < u) = \frac{1}{[x]} \#\{m \leqslant x, f_x(m) < u\}$$

silpnam konvergavimui. Čia $f_x(m)$ yra šeima ($x \geqslant 2$) stipriai adityviųjų funkcijų, kurioms $f_x(p) \in \{0, 1\}$ visiems pirminiams p.