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Specialization of loop rules of a sequent calculus of intuitionistic temporal logic with time gaps

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1. Introduction

A traditional calculus of intuitionistic first-order logic contains the rules $(\supset\rightarrow)$ and $(\forall\rightarrow)$ the main formulas of which are duplicated in the premises of these rules. It is inconvenient for computer-based proof search.

As it was discovered by Vorob'ev [7,8], there exists a sequent proof search algorithm allowing to avoid duplication of the main formula $E \supset B$, where E is an atomic formula, in the rule $(\supset\rightarrow)$. This idea was extended in the later works [3, 4, 5].

In the paper, we give an intuitionistic first-order sequent calculus LBJ with the rules $(\supset\rightarrow)$, $(\forall\rightarrow)$, and present a partial solution to the problem arising owing to the rules $(\supset\rightarrow)$, $(\forall\rightarrow)$. As far as the rule $(\supset\rightarrow)$ is concerned, we mainly base our investigation on the work [4].

2. Calculus LBJ

Constructing the calculus LBJ, we use an intuitionistic variant of a sequent calculus—LJ—without the structural rules:

1. Axioms: $\Gamma, E \rightarrow E$; $\Gamma, \mathcal{F} \rightarrow \Delta$
2. Derivation rules:

$$\frac{\Gamma, A \rightarrow B}{\Gamma \rightarrow A \supset B}\ (\rightarrow\supset) \qquad \frac{A \supset B, \Gamma \rightarrow A;\ B, \Gamma \rightarrow \Delta}{A \supset B, \Gamma \rightarrow \Delta}\ (\supset\rightarrow)$$

$$\frac{\Gamma \rightarrow A;\ \Gamma \rightarrow B}{\Gamma \rightarrow A \wedge B}\ (\rightarrow\wedge) \qquad \frac{A, B, \Gamma \rightarrow \Delta}{A \wedge B, \Gamma \rightarrow \Delta}\ (\wedge\rightarrow)$$

$$\frac{\Gamma \rightarrow A \text{ or } \Gamma \rightarrow B}{\Gamma \rightarrow A \vee B}\ (\rightarrow\vee) \qquad \frac{A, \Gamma \rightarrow \Delta;\ B, \Gamma \rightarrow \Delta}{A \vee B, \Gamma \rightarrow \Delta}\ (\vee\rightarrow)$$

$$\frac{\Gamma \rightarrow A(b)}{\Gamma \rightarrow \forall x A(x)}\ (\rightarrow\forall) \qquad \frac{A(t), \forall x A(x), \Gamma \rightarrow \Delta}{\forall x A(x), \Gamma \rightarrow \Delta}\ (\forall\rightarrow)$$

$$\frac{\Gamma \rightarrow A(t)}{\Gamma \rightarrow \exists x A(x)}\ (\rightarrow\exists) \qquad \frac{A(b), \Gamma \rightarrow \Delta}{\exists x A(x), \Gamma \rightarrow \Delta}\ (\exists\rightarrow)$$

Here: $\mathcal{F}$ denotes 'false'; E denotes an atomic formula; A, B denote arbitrary formulae; $\Delta \in \{\emptyset, D\}$ (D is an arbitrary formula); Γ denotes a finite, possibly empty, multiset of formulae; x

denotes a bound variable; t denotes a term (expressions in which bound variables occur are not assumed to be terms); in the $(\forall \rightarrow)$, $(\rightarrow \exists)$ rules, $A(x)$ is obtained by substituting x for at least one occurrence of t in $A(t)$; b denotes a free variable which does not occur in conclusions of the rules $(\rightarrow \forall)$, $(\exists \rightarrow)$, and $A(x)$ is obtained, in these rules, by substituting x for every occurrence of b in $A(b)$; we use different letters to denote free and bound variables so that a variable can be only free or only bound; we do not have rules for negation: $\neg A =_{def} A \supset \mathcal{F}$.

The calculus LBJ is obtained from the calculus LJ by adding the following rules for handling the operators $\Box, \Diamond, \circ$:

$$\frac{\Gamma \rightarrow A}{\Pi, \circ\Gamma \rightarrow \circ A} (\circ_1) \qquad \frac{\Gamma \rightarrow}{\Pi, \circ\Gamma \rightarrow \Delta} (\circ_2)$$

$$\frac{\Box\Gamma, C \rightarrow D}{\Pi, \Box\Gamma, \Diamond C \rightarrow \Diamond D} (\Diamond_1) \qquad \frac{\Box\Gamma, C \rightarrow}{\Pi, \Box\Gamma, \Diamond C \rightarrow \Delta} (\Diamond_2)$$

$$\frac{\Gamma \rightarrow A}{\Gamma \rightarrow \Diamond A} (\rightarrow \Diamond_1) \qquad \frac{\Gamma \rightarrow \circ\Diamond A}{\Gamma \rightarrow \Diamond A} (\rightarrow \Diamond_2)$$

$$\frac{A, \Gamma \rightarrow \Delta; \circ\Diamond A, \Gamma \rightarrow \Delta}{\Diamond A, \Gamma \rightarrow \Delta} (\Diamond \rightarrow) \qquad \frac{\Box\Gamma \rightarrow A}{\Pi, \Box\Gamma \rightarrow \Box A} (\Box)$$

$$\frac{A, \circ\Box A, \Gamma \rightarrow \Delta}{\Box A, \Gamma \rightarrow \Delta} (\Box \rightarrow) \qquad \frac{\Gamma \rightarrow A; \Gamma \rightarrow \circ\Box A}{\Gamma \rightarrow \Box A} (\rightarrow \Box)$$

Here: Γ, Δ, A are as in LJ; if $\Gamma = A_1, A_2, \ldots, A_n$, then $\sigma\Gamma = \sigma A_1, \sigma A_2, \ldots, \sigma A_n$, where $\sigma \in \{\Box, \circ\}$; C, D denote arbitrary formulae; Π denotes an arbitrary finite, possibly empty, multiset of formulae. The rule $(\rightarrow \Box)$ corresponds to the weak induction axiom: $(A \wedge \circ\Box A) \supset \Box A$.

The definition of derivation is common (see e.g. [6]). We denote derivations and heights of the derivations by V and $h(V)$ (V may be with an index, an asterisk, one or two primes) respectively.

3. Specialization of the $(\supset\rightarrow)$ rule

A calculus LBJT is obtained from the calculus LBJ by substituting the following

$$\frac{B, E, \Gamma \rightarrow \Delta}{E \supset B, E, \Gamma \rightarrow \Delta} (E \supset\rightarrow) \qquad \frac{\alpha C \supset (\alpha D \supset B), \Gamma \rightarrow \Delta}{\alpha(C \wedge D) \supset B, \Gamma \rightarrow \Delta} (\alpha \supset\rightarrow)$$

$$\frac{\beta C \supset B, \beta D \supset B, \Gamma \rightarrow \Delta}{\beta(C \vee D) \supset B, \Gamma \rightarrow \Delta} (\beta \supset\rightarrow) \qquad \frac{C, D \supset B, \Gamma \rightarrow D; B, \Gamma \rightarrow \Delta}{(C \supset D) \supset B, \Gamma \rightarrow \Delta} (\supset\supset\rightarrow)$$

$$\frac{\Gamma \rightarrow \circ A'; B, \Gamma \rightarrow \Delta}{\circ A' \supset B, \Gamma \rightarrow \Delta} (\circ \supset\rightarrow) \qquad \frac{\forall x(A(x) \supset B), \Gamma \rightarrow \Delta}{\exists x A(x) \supset B, \Gamma \rightarrow \Delta} (\exists \supset\rightarrow)$$

$$\frac{A^* \supset B, \Gamma \rightarrow A^*; B, \Gamma \rightarrow \Delta}{A^* \supset B, \Gamma \rightarrow \Delta} (* \supset\rightarrow)$$

Here: E denotes an atomic formula; $\alpha = \alpha_1\alpha_2\ldots\alpha_n$, where $n \geqslant 0$, $\alpha_i \in \{\Box, \circ, \forall x_i\}$, x_i denotes any bound variable, $1 \leqslant i \leqslant n$; $\beta = \beta_1\beta_2\ldots\beta_m$, $\beta_j \in \{\exists x_j\}$, where $m \geqslant 0$, x_j denotes any bound variable, $1 \leqslant j \leqslant m$; $A' \neq \alpha(C \wedge D)$, where α is defined as above; $A^* \in \{oE, o(C \vee D), o(C \supset D), o\exists xC, o\forall xC, o'(C \wedge D)\}$, where $o = o_1o_2\ldots o_n$, where $n \geqslant 1$, $o_i \in \{\Box, \Diamond, \forall x_i\}$, x_i denotes any bound variable, $1 \leqslant i \leqslant n$; $o' = \Diamond$ or $o' = o'_1o'_2\ldots o'_m$, where $m \geqslant 2$, $o'_j \in \{\circ, \Box, \Diamond, \exists x_j, \forall x_j\}$, x_j denotes any bound variable, $1 \leqslant j \leqslant m$, $o'_1 \neq \circ$, and $\Diamond$ occurs in o'.

In some proofs, we use complexity of formulae which is denoted by $\mathcal{G}$ and defined as follows:

1) $\mathcal{G}(E) = 0$, $\mathcal{G}(\mathcal{F}) = 0$;

2) $\mathcal{G}(B \vee D) = \mathcal{G}(B \supset D) = \mathcal{G}(B) + \mathcal{G}(D) + 1$;

3)$\mathcal{G}(B \wedge D) = \mathcal{G}(B) + \mathcal{G}(D) + 2$;

4)$\mathcal{G}(\forall x B(x)) = \mathcal{G}(B(t)) + 1$, where t denotes an arbitrary term, $B(t)$ is obtained by substituting t for every occurrence of x in $B(x)$;

5) $\mathcal{G}(\exists x B(x)) = \mathcal{G}(B(t)) + 2$, where t denotes an arbitrary term, $B(t)$ is obtained by substituting t for every occurrence of x in $B(x)$;

6) $\mathcal{G}(\sigma B) = \mathcal{G}(B) + 1$, where $\sigma \in \{\Box, \Diamond\}$;

7)$\mathcal{G}(\circ A) = \mathcal{G}(A)$, if $A = \Box B$ or $A = \Diamond B$, otherwise $\mathcal{G}(\circ A) = \mathcal{G}(A) + 1$; here: E denotes an atomic formula; B, D denote arbitrary formulae; x denotes a bound variable.

Lemma 3.1. *Let* $(W \rightarrow)$ *denote the structural antecedent—and* $(\rightarrow W)$ *succedent—weakening rule, then*

1) if $(LBJT + (W \rightarrow)) \vdash^V S$*, then there exists a derivation* V' *such that* $LBJT \vdash^{V'} S$ *and* $h(V') \leqslant h(V)$.

2) if $(LBJT + (\rightarrow W)) \vdash^V S$*, then there exists a derivation* V' *such that* $LBJT \vdash^{V'} S$ *and* $h(V') \leqslant h(V)$;

Proof. See [1].

Lemma 3.2. *Let* $(i) \in \{(\gamma \supset\rightarrow), (\rightarrow\supset), (\rightarrow \wedge), (\wedge \rightarrow), (\vee \rightarrow), (\rightarrow \forall), (\forall \rightarrow), (\exists \rightarrow), (\rightarrow \Box), (\Diamond \rightarrow)\}$*, where* $\gamma \in \{E, \alpha, \beta, \supset, \circ, \exists, *\}$ *and* S *be a sequent having the shape of the conclusion of the rule* (i)*. Let* S' *be the sequent having the shape of a premise of the same rule* (i)*, we take only the right premise for* $(i) = (\circ \supset\rightarrow)$, $(i) = (\supset\supset\rightarrow)$, $(i) = (* \supset\rightarrow)$ *and only the left premise for* $(i) = (\rightarrow \Box)$, $(i) = (\Diamond \rightarrow)$*, then if* $LBJT \vdash^V S$*, then there exists a derivation* V' *such that* $LBJT \vdash^{V'} S'$ *and* $h(V') \leqslant h(V)$.

Proof. See [1, 4].

Lemma 3.3. *The rules* $(\Box \rightarrow)$, $(\rightarrow \Box)$, $(\Diamond \rightarrow)$ *are invertible in LBJT.*

Proof. See [1].

Lemma 3.4. *A sequent of the type* $\Gamma, A \rightarrow A$*, where* A *is an arbitrary formula, is derivable in LBJT.*

Proof. The Lemma is proved by complexity of formulae.

Lemma 3.5. *The rule*

$$\frac{\Gamma \rightarrow B; \Gamma, D \rightarrow \Delta}{\Gamma, B \supset D \rightarrow \Delta}$$

is admissible in LBJT.

Proof. The lemma is proved by induction on the ordered pair $\langle \mathcal{G}(B), h \rangle$, where $\mathcal{G}(B)$ denotes the complexity of the formula B, and h denotes the height of derivation of the first premise. For the details, see [4].

Lemma 3.6. *The rule*

$$\frac{\Gamma, (C \supset B) \supset D \to \Delta}{\Gamma, C, B \supset D, B \supset D \to \Delta}$$

is admissible in LBJT.

Proof. See [4].

Lemma 3.7. *The rule*

$$\frac{\circ B \supset D, \Gamma \to \circ B;\, D, \Gamma \to \Delta}{\circ B \supset D, \Gamma \to \Delta}$$

where $B \neq \alpha(B_1 \wedge B_2)$, *where* α *is as in the* $(\alpha \supset\to)$ *rule, is admissible in LBJT.*

Proof. The lemma is proved by induction on the height of the derivation of the first premise.

Theorem 3.8. *The contraction rule*

$$\frac{A, A, \Gamma \to \Delta}{A, \Gamma \to \Delta}$$

is admissible in LBJT

Proof. The lemma is proved by the ordered pair $\langle \mathcal{G}(A), h(V) \rangle$, where $\mathcal{G}(A)$ denotes the complexity of the contraction formula, and $h(V)$ denotes height of derivation of the conclusion of the contraction rule. For the details, see [4, 1].

Lemma 3.9. *The rule*

$$\frac{A \supset B, \Gamma \to A;\, B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}$$

is admissible in LBJT.

Proof. We weaken the second premise with $A \supset B$, use Lemma 3.5, and contract the formula $A \supset B$.

It follows from the previous lemma that all LBJ rules are admissible in LBJT. As weakening, contraction and cut are admissible in LBJ (see [1]), it is straightforward to prove that all LBJT rules are admissible in LBJ. So we have the following theorem:

Lemma 3.10. *The calculi LBJ and LBJT are equivalent.*

The following theorem follows from this theorem:

Theorem 3.11. *The cut rule is admissible in LBJT.*

4. Specialization of the $(\forall \rightarrow)$ rule

Let us assume that Γ^* is such a multiset of formulae in which $\supset$ and formulas of the type $\circ A'$, A^* do not occur in the left scope of $\supset$, where $\circ A'$ and A^* are as in the $(* \supset\rightarrow)$, $(\circ \supset\rightarrow)$ rules; Δ^* is an empty set or a formula in which $\wedge$, $\forall$, $\Box$ can occur only in the left scope of $\supset$, and $\supset$, in whose left scope occurs at least one $\supset$ or a formula of the type $\circ A'$ or A^* (A' and $\circ A^*$ are as above), does not occur in the left scope of $\supset$. Further, let us assume that if an atomic formula occurs in the left scope of $\supset$ in Γ^* or Δ^*, then this atomic formula can occur only as a subformula in an implication formula in Γ^* or Δ^*. The sequent $\Gamma^* \rightarrow \Delta^*$ is called a σ-sequent.

First notice that if a conclusion of any LBJT rule is a σ-sequent, then the premise of this rule is a σ-sequent, as well.

Lemma 4.1. *If a σ-sequent is derivable in LBJT and $\Gamma \neq \emptyset$, then there exists a formula G in Γ such that $LBJT \vdash G \rightarrow \Delta$*

Proof. By induction on height of derivation of $\Gamma \rightarrow \Delta$

Theorem 4.2. *A σ-sequent $\forall x A(x), \Gamma \rightarrow \Delta$ is derivable in LBJT iff, for some term t, the sequent $A(t), \Gamma \rightarrow \Delta$ is derivable in LBJT.*

Proof. The theorem is proved by height of derivation of $\forall x A(x), \Gamma \rightarrow \Delta$, using Lemma 4.1.

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Intuicionistinės laiko logikos su laiko tarpais sekvencinio skaičiavimo ciklinių taisyklių specializacija

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Straipsnyje nagrinėjama intuicionistinės laiko logikos su laiko tarpais sekvencinio skaičiavimo ciklinių taisyklių specializacijos problema. Ciklinė implikacijos antecedente taisyklė keičiama keliomis kitomis taisyklėmis, dalinai išsprendžiant šios taisyklės cikliškumo problemą. Nurodomos sąlygos, kurioms esant galima atsisakyti visuotinumo kvantoriaus antecedente taisyklės pagrindinės formulės dubliacijos, įrodomų sekvencijų klasei išliekant nepakitusiai.