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Formula for Dupin cyclidic cube and Miquel point*

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Abstract. Dupin cyclides are surfaces conformally equivalent to a torus, a circular cone, or a cylinder. Their patches admit rational bilinear quaternionic Bézier (QB) parametrizations and are used in geometric design and architecture. Dupin cyclidic cubes are a natural trivariate generalization of Dupin cyclide patches. In this article, we derive explicit formulas for control points and weights of rational 3-linear QB parametrizations of Dupin cyclidic cubes and relate them with classical Miquel point construction.

Keywords: Dupin cyclide; Dupin cyclidic cube; quaternionic-Bézier formula

AMS Subject Classification: 65D17, 53A70

Introduction

Dupin cyclides, i.e., surfaces conformally equivalent to a torus, circular cone, or cylinder, have versatile applications in geometric design and architecture. They have circular curvature lines. Their patches bounded by 4 circles, which are curvature lines, allow rational bilinear quaternionic Bézier (QB) parametrizations and offer significant advantages in modeling complex shapes. Building on the foundational concepts of Dupin cyclide principal patches, Dupin cyclidic (DC) cubes represent a natural trivariate extension. We can effectively model these higher-dimensional structures by employing rational trilinear QB parametrizations.

This paper presents explicit formulas for the control points and weights necessary for the rational trilinear QB parametrizations of DC cubes. Additionally, we establish

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a connection between these parametrizations and the classical Miquel point construction. The quaternionic representations of DC cubes were recently used in [1], and the present paper supports these results. Using geometric constructive derivation, a preliminary QB formula for DC cubes was given in [2].

1 Quaternions and inversions

The algebra of quaternions $\mathbb{H}$ is the real non-commutative algebra generated by $\{\mathbf{i},\mathbf{j},\mathbf{k}\}$ satisfying the product rules $\mathbf{i}^2=\mathbf{j}^2=\mathbf{k}^2=\mathbf{ijk}=-1$. It is a 4-dimensional real vector space with the standard basis $\{1,\mathbf{i},\mathbf{j},\mathbf{k}\}$. For a quaternion q written in the algebraic form $q=r+x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, define the real part $\mathrm{Re}(q)=r$, the imaginary part $\mathrm{Im}(q)=q-\mathrm{Re}(q)$, the conjugate $\bar{q}=\mathrm{Re}(q)-\mathrm{Im}(q)$, and the norm $|q|=\sqrt{q\bar{q}}$. The algebra $\mathbb{H}$ is also a division ring, meaning that every non-zero element is invertible. If $q\neq 0$, its inverse is $q^{-1}=\bar{q}/|q|^2$. The properties of conjugation and norm are the same as those of complex numbers, but care must be taken to account for non-commutativity when permuting product elements; e.g., $\overline{qp}=\bar{p}\bar{q}$. We refer to [2, 3] for further details about quaternionic products and their relation to the standard dot and cross products in $\mathbb{R}^3$. The Euclidean space $\mathbb{R}^3$ here is identified with the space of imaginary quaternions $\mathrm{Im}\mathbb{H}=\{q\in\mathbb{H}\mid \mathrm{Re}(q)=0\}$.

Since we are dealing with objects composed of circles and lines in $\mathbb{R}^3$, it is natural to use inversion transformations that preserve the set of circles and lines and angles between crossing curves, known as conformal transformations. In the quaternionic framework, an inversion Inv_q^r with respect to a sphere of center $q\in\mathrm{Im}\mathbb{H}$ and radius $r>0$ can be written explicitly as

$$\mathrm{Inv}_q^r(p)=q-r^2(p-q)^{-1}\in\mathrm{Im}\mathbb{H}$$

for all $p\in\mathrm{Im}\mathbb{H}$. On the compactified space $\widehat{\mathbb{R}}^3=\mathbb{R}^3\cup\{\infty\}$, which is identified with $\mathrm{Im}\widehat{\mathbb{H}}=\mathrm{Im}\mathbb{H}\cup\{\infty\}$, Inv_q^r is an involution transformation mapping the center q to ∞ and vice versa. The group generated by inversions is called the group of Möbius transformations. Euclidean similarities are particular cases of Möbius transformations; see [3] for more details.

2 Rational quaternionic Bézier curves

For two quaternions U,W, define the quaternionic fraction $\frac{U}{W}=UW^{-1}$ if $W\neq 0$ and $\frac{U}{W}=\infty$ if $W=0$. A rational quaternionic Bézier (QB) curve $C(t)$ of degree n is defined by the following data:

- Control points $p_i\in\mathbb{H}$, $i=0,\dots,n$;
- Weights $w_i\in\mathbb{H}$, $i=0,\dots,n$;

such that

$$C(t)=\frac{\sum_{i=0}^n p_iw_iB_i^n(t)}{\sum_{i=0}^n w_iB_i^n(t)},$$

where $B_i^n(t) = \binom{n}{i}(1-t)^{n-i}t^i$ are Bernstein basis polynomials. The matrix

$$\begin{pmatrix} u_i \\ w_i \end{pmatrix}_{i=0\ldots n} = \begin{pmatrix} p_i w_i \\ w_i \end{pmatrix}_{i=0\ldots n}$$

is called the homogeneous representation of the curve $C(t)$. We will be interested in the linear case ($n = 1$).

Remark 1. QB formulas are preserved by inversions in the following sense: an inversion Inv_q^r maps a QB formula with homogeneous control points (u_i, w_i) to a QB formula with homogeneous control points (u_i', w_i') such that

$$u_i' = qu_i - (r^2 + q^2)w_i, \quad w_i' = u_i - qw_i. \tag{1}$$

To model curves in $\widehat{\mathbb{R}}^3$, let us standardize the condition for an arbitrary pair (U, W) of quaternions to define a point $UW^{-1} \in \widehat{\mathbb{R}}^3$. By identifying $\mathbb{H}^2$ with $\mathbb{R}^8$, define the quadratic form S in $\mathbb{R}^8$ by

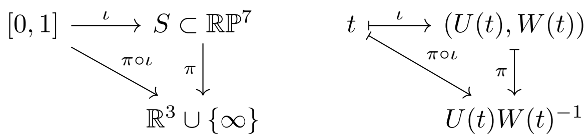
$$S(u, w) = \frac{u\bar{w} + w\bar{u}}{2}, \quad (u, w) \in \mathbb{H}^2. \tag{2}$$

The quadric in $\mathbb{R}P^7$ (real projectivization $\mathbb{H}^2$) defined by $S(u, w) = 0$ is called the *Study quadric*, which we denote by S as well. Let $\pi : \mathbb{H}^2 \to \mathbb{H} \cup \infty$ such that $\pi(u, w) = uw^{-1}$. The following is straightforward:

Lemma 1. *$\pi(u, w) = uw^{-1} \in \widehat{\mathbb{R}}^3$ if and only if $(u, w) \in S$.*

To design a QB curve in $\widehat{\mathbb{R}}^3$, we follow the following routines:

- The control points p_i are contained in $\widehat{\mathbb{R}}^3$;
- The homogeneous control points $(p_i w_i, w_i)$ are contained in the Stdudy quadric;
- Consider the pair $(U(t), W(t))$ as a standard Bézier curve with real weights in the Study quadric. Then, apply the projection π to get a curve in $\widehat{\mathbb{R}}^3$; see the diagram below.



Example 1. A circular arc with endpoints p_0, p_1 and a tangent vector v_1 at p_0 can be parametrized using the QB formula:

$$\begin{pmatrix} u_0 & u_1 \\ w_0 & w_1 \end{pmatrix} = \begin{pmatrix} p_0 & p_1(p_1 - p_0)^{-1}v_1 \\ 1 & (p_1 - p_0)^{-1}v_1 \end{pmatrix}. \tag{3}$$

Such formulas can be found in [1, 2, 3]. Note that if $p_0 = \infty$, then we have a semi-line starting from p_1 in the direction of v_1. This semi-line can be parametrized using the QB formula:

$$\begin{pmatrix} u_0 & u_1 \\ w_0 & w_1 \end{pmatrix} = \begin{pmatrix} 1 & -p_1 v_1 \\ 0 & -v_1 \end{pmatrix}. \tag{4}$$

A reparametrization of the arc is obtained if we multiply w_1 by a constant $\lambda > 0$. If $\lambda < 0$, then a parametrization of the complementary arc is obtained. If the arc is defined by the endpoints p_0, p_1 and a point q on the complementary arc, then the weights can be assigned as

$$w_0 = (q - p_0)^{-1}, \qquad w_1 = (p_1 - q)^{-1}.$$

3 QB formulas for Dupin cyclide principal patches

A bivariate generalization of the QB formula of circular arcs yields a so-called Dupin cyclide principal patch. They are quad patches bounded by 4 circular arcs intersecting orthogonally at the corner points. Note that the 4 corner points are always cocircular. This circularity condition can be interpreted in terms of cross-ratio. The cross-ratio between points $p_0, p_1, p_2, p_3 \in \mathrm{Im}\mathbb{H}$ is defined as

$$\mathrm{cr}(p_0, p_1, p_2, p_3) = (p_0 - p_1)(p_1 - p_2)^{-1}(p_2 - p_3)(p_3 - p_0)^{-1},$$

whenever the product is well-defined.

Remark 2. Four points $p_0, p_1, p_2, p_3 \in \mathrm{Im}\mathbb{H}$ are cocircular if and only if their cross-ratio is real; see [3, Lemma 2.3].

A Dupin cyclide principal patch is uniquely determined by its four cocircular corner points and tangent vectors v_1, v_2 at one corner point. The tangent vectors at other points are obtained by reflection along the respective edges. The following results about QB representation of principal patches follow from [1, Theorem 2.2]. Let a Dupin cyclide principal patch be defined by cocircular corner points p_0, p_1, p_2, p_3 and orthogonal tangent vectors v_1 and v_2 at p_0 and let $v_3 = v_1 v_2$. Then, this patch can parametrized using the bilinear QB formula with the following homogeneous representation:

(i) $p_0 = \infty$ and p_1, p_2, p_3 are collinear, $p_1 \neq p_2$, then the first control point is $(u_0, w_0) = (1, 0)$, and the others are $(p_i w_i, w_i)$ such that

$$w_1 = -v_1, \quad w_2 = -v_2, \quad w_3 = (p_1 - p_2)v_3. \tag{5}$$

(ii) all control points are finite, only p_1 and p_2 may coincide with p_3, then

$$w_0 = 1, \quad w_1 = q_{01}v_1, \quad w_2 = q_{02}v_2, \quad w_3 = q_{03}(q_{01} - q_{02})v_3, \tag{6}$$

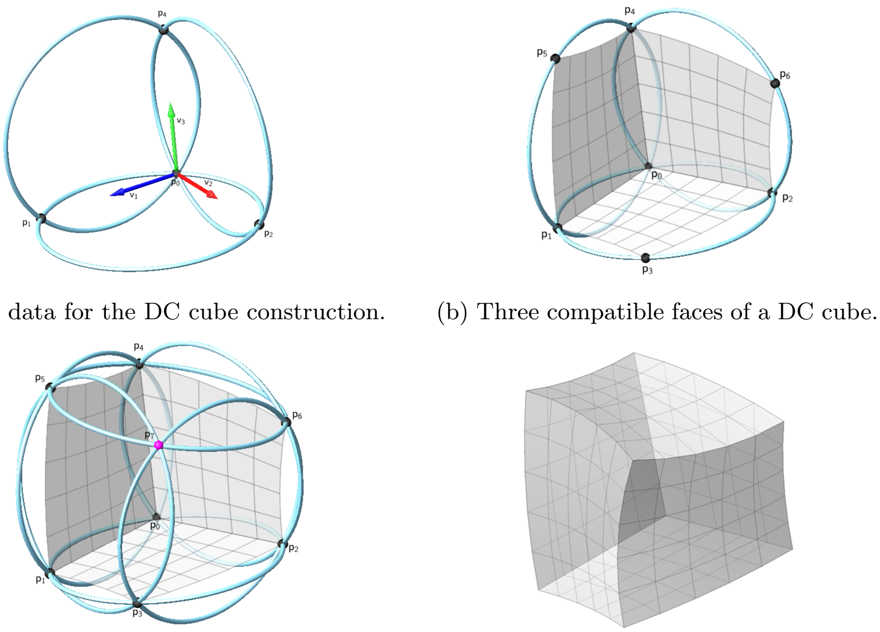
where $q_{0i} = (p_i - p_0)^{-1}$ for $i = 1, 2, 3$.

4 QB formulas for Dupin cyclidic cubes

A DC cube is a 3-linear rational quaternionic map

$$F : [0,1]^3 \to \widehat{\mathbb{R}}^3, \qquad F = UW^{-1}, \quad U, W \in \mathbb{H}[s, t, u],$$

such that all three partial derivatives $\partial_s F$, $\partial_t F$, $\partial_u F$ are mutually orthogonal, and the Jacobian $\mathrm{Jac}(F)$ is not identically zero. As we will investigate, DC cubes can be expressed using QB formulas with their 8 corner points as control points and quaternionic weights. We refer to [1] for more details about a DC cube construction. We will highlight the following essential properties:



(a) Initial data for the DC cube construction. (b) Three compatible faces of a DC cube.

(c) The Miquel point p_7. (d) The resulting DC cube.

Fig. 1. Steps of a DC cube construction.

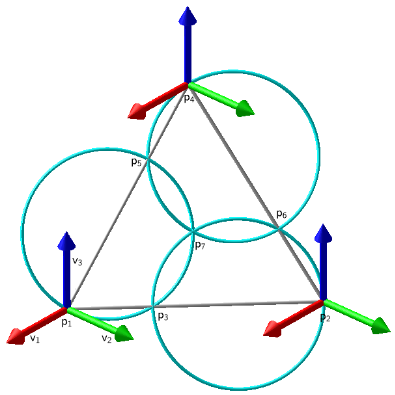
- The 8 control points of a DC cube are always cospherical or coplanar.
- A DC cube is uniquely defined by its 3 adjacent faces at one corner point, see Fig. 1(b).
- The condition to make a compatible 3 adjacent faces at a corner point, say p_0, can be defined by the data: an orthonormal frame v_1, v_2, v_3 at p_0, corner points p_1, p_2, p_4; a point p_{i+j} on the circle $(p_0 p_i p_j)$, $i, j \in \{1, 2, 4\}$.
- The last control point p_7 can be derived using Miquel theorem about the intersection of 3 circles; see Fig. 2 or the explanation in [1] for more details.

This paper's derivation first constructs a DC cube using one corner point on infinity and then applies inversions to derive the QB formula for general DC cubes.

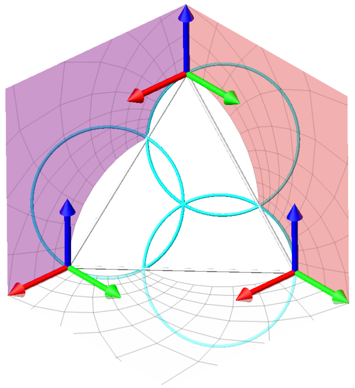
Theorem 1. *Let a DC cube be defined by 8 corner points $p_0 = \infty$, $p_1, p_2, p_4 \in \mathrm{Im}\mathbb{H}$, p_3, p_5, p_6 on the lines $(p_1p_2), (p_1p_4), (p_2p_4)$ respectively, with the associated Miquel point p_7 and an orthonormal frame $\{v_1, v_2, v_3 = v_1v_2\} \subset \mathrm{Im}\mathbb{H}$ at p_0. Then, we can parametrize it using a trivariate QB parametrization with the following homogeneous control points*:

$$\begin{pmatrix} 1 & -p_1v_1 & -p_2v_2 & p_3(p_1-p_2)v_3 & -p_4v_3 & p_5(p_4-p_1)v_2 & p_6(p_2-p_4)v_1 & p_7w_7 \\ 0 & -v_1 & -v_2 & (p_1-p_2)v_3 & -v_3 & (p_4-p_1)v_2 & (p_2-p_4)v_1 & w_7 \end{pmatrix},$$

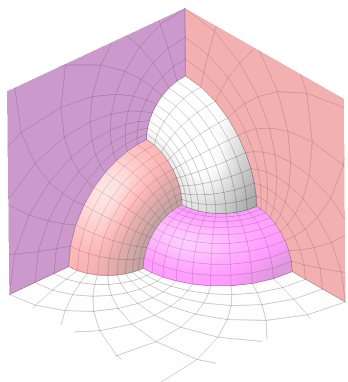
where w_7 has the following equivalent expressions



(a) Initial data for a DC cube with one corner point on infinity.



(b) Three compatible faces of a DC cube with a common intersection point on infinity.



(c) The resulting 6 faces of the DC cube from 3 compatible faces.

Fig. 2. DC cube construction steps with one control point on infinity.

$$w_7 = (p_7 - p_1)^{-1}(p_4 - p_1)(p_3 - p_5)(p_1 - p_2) \tag{7}$$
$$= (p_7 - p_2)^{-1}(p_1 - p_2)(p_6 - p_3)(p_2 - p_4) \tag{8}$$
$$= (p_7 - p_4)^{-1}(p_2 - p_4)(p_5 - p_6)(p_4 - p_1). \tag{9}$$

Proof. Let f_{0123}, f_{0415} and f_{0246} be the initial three faces of the DC cube meeting at $p_0 = \infty$; see Fig. 2(b). Using formula (5), we obtain the presented formula for w_i, $i = 0, \dots, 6$. Note that the frames at p_1, p_2, and p_4 for the DC cube are the same. On the face f_{4567}, we compute the weights using formula (6). This gives

$$w'_4 = 1, \quad w'_5 = (p_5 - p_4)^{-1}v_1, \quad w'_6 = (p_6 - p_4)^{-1}v_2,$$
$$w'_7 = (p_7 - p_4)^{-1}(p_6 - p_4)^{-1}(p_6 - p_5)(p_5 - p_4)^{-1}v_3.$$

To get the compatibility at p_4, we need to multiply such weights with $-v_3$. This gives

$$w''_4 = -v_3, \quad w''_5 = (p_5 - p_4)^{-1}v_2, \quad w''_6 = -(p_6 - p_4)^{-1}v_1,$$
$$w''_7 = (p_7 - p_4)^{-1}(p_6 - p_4)^{-1}(p_6 - p_5)(p_5 - p_4)^{-1}.$$

To get the compatibility at p_5 and p_6, we multiply w_5'' by $\lambda_1 = (p_4 - p_1)(p_5 - p_4)$ and w_6'' by $\lambda_2 = -(p_2 - p_4)(p_6 - p_4)$. Note that λ_1 and λ_2 are real because the points p_1, p_4, p_5 and similarly p_2, p_4, p_6 are collinear. Hence, a reparametrization of the face f_{4567} using $w_4'' = w_4$, $\lambda_1 w_5'' = w_5$, $\lambda_2 w_6'' = w_6$ and $\lambda_1\lambda_2 w_7'' = w_7$, which is the compatible weight at p_7. In the product $\lambda_1\lambda_2 w_7''$, the factors $p_5 - p_4$ and $p_6 - p_4$ of λ_1 and λ_2 will be eliminated, giving the formula (9) for w_7. By studying the compatibility similarly on the faces f_{1357} and f_{2637}, we obtain alternative formulas for w_7 in (7) and (8). It follows from the compatibility lemma [1, Lemma 3.3] that the 3 found weights have to coincide, giving a compatible parametrization of the DC cube. □

Corollary 1. *The Miquel point p_7 can be expressed as*

$$\begin{aligned} p_7 &= p_1 + A(A - B)^{-1}(p_2 - p_1) && (10)\\ &= p_2 + B(B - C)^{-1}(p_4 - p_2) && (11)\\ &= p_4 + C(C - A)^{-1}(p_1 - p_4), && (12) \end{aligned}$$

where A, B, C are the right-quaternionic factors of w_7, namely

$$\begin{aligned} A &= (p_4 - p_1)(p_3 - p_5)(p_1 - p_2),\\ B &= (p_1 - p_2)(p_6 - p_3)(p_2 - p_4),\\ C &= (p_2 - p_4)(p_5 - p_6)(p_4 - p_1). \end{aligned}$$

Proof. From (7) and (8), we have $w_7 = (p_7 - p_1)^{-1}A = (p_7 - p_2)^{-1}B$. This implies

$$\begin{aligned} BA^{-1} &= (p_7 - p_2)(p_7 - p_1)^{-1}\\ &= (p_7 - p_1 + p_1 - p_2)(p_7 - p_1)^{-1}\\ &= 1 + (p_1 - p_2)(p_7 - p_1)^{-1}. \end{aligned}$$

Hence $(p_7 - p_1)^{-1} = (p_1 - p_2)^{-1}(BA^{-1} - 1) = (p_1 - p_2)^{-1}(B - A)A^{-1}$, i.e, $p_7 - p_1 = A(B - A)^{-1}(p_1 - p_2) = A(A - B)^{-1}(p_2 - p_1)$. We obtain (10) by adding p_1 on both sides. The expressions (11) and (12) can be obtained similarly by considering other pairs of expressions for w_7. □

We apply inversions to relate the formula in Theorem 1 to a general formula for DC cubes with finite control points.

Theorem 2. *Let a DC cube be defined by 8 cospherical corner points $p_0, p_1, p_2, p_4 \in \mathrm{Im}\mathbb{H}$, p_3 on the circle $(p_0p_1p_2)$, p_5 on the circle $(p_0p_1p_4)$, p_6 on the circle $(p_0p_2p_4)$, the associated Miquel point p_7, and an orthonormal frame $\{v_1, v_2, v_3 = v_1v_2\} \subset \mathrm{Im}\mathbb{H}$ at p_0; see Fig. 1. Let $q_{0i} = (p_i - p_0)^{-1}$ for $i = 1, \dots, 7$. Then, this cube can be parametrized using the homogeneous control points (p_iw_i, w_i), $i = 0, \dots, 7$, where*

$$\begin{gathered} w_0 = 1, \quad w_1 = q_{01}v_1, \quad w_2 = q_{02}v_2, \quad w_4 = q_{04}v_3,\\ w_3 = q_{03}(q_{01} - q_{02})v_3, \quad w_5 = q_{05}(q_{04} - q_{01})v_2, \quad w_6 = q_{06}(q_{02} - q_{04})v_1,\\ w_7 = -q_{07}(q_{07} - q_{01})^{-1}(q_{04} - q_{01})(q_{03} - q_{05})(q_{01} - q_{02}). \end{gathered}$$

Proof. This is equivalent to the formula in Theorem 1 using inversions as addressed in Remark 1. For instance, let us consider the derivation of w_7. We apply first $\mathrm{Inv}^1_{p_0}$ and all the control points are transformed to $p'_0 = \infty$ and $p'_i = p_0 - q_{0i}$, $i = 1, \dots, 7$. By Theorem 1, we have $w'_7 = (q_{07} - q_{01})^{-1}(q_{04} - q_{01})(q_{03} - q_{05})(q_{01} - q_{02})$. Hence, by applying the same inversion, we obtain $w_7 = (p'_7 - p_0)w'_7 = -q_{07}w'_7$. This coincides with the displayed formula for w_7. □

The following result follows from Corollary 1 by applying inversions.

Corollary 2. *With the notations in Theorem 2, the 8th control point p_7 of the DC cube, analogue of the Miquel point on the plane, can be expressed as*

$$p_7 = p_0 + \left[q_{01} + A'(A' - B')^{-1}(q_{02} - q_{01})\right]^{-1}, \tag{13}$$

where

$$A' = (q_{04} - q_{01})(q_{03} - q_{05})(q_{01} - q_{02}),$$
$$B' = (q_{01} - q_{02})(q_{06} - q_{03})(q_{02} - q_{04}).$$

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REZIUMĖ

Dupino ciklidinio kubo formulė ir Miquelio taškas

J.M. Menjanahary, R. Krasauskas

Dupino ciklidės yra paviršiai, konformiškai ekvivalentūs torui, apskritiminiam kūgiui arba cilindrui. Jų skiautės parametrizuojamos bitiesinėmis kvaternioninėmis Bézier (KB) formulėmis ir yra naudojamos geometriniame modeliavime ir architektūroje. Dupino ciklidiniai kubai yra natūralus trimatis Dupino ciklidžių skiaučių apibendrinimas. Šiame straipsnyje mes pateikiame Dupino ciklidinių kubų racionalių 3-tiesinių KB reprezentacijų kontrolinių taškų ir svorių formules, ir susiejame jas su klasikine Miquelio taško konstrukcija.

Raktiniai žodžiai: Dupino ciklidė; Dupino ciklidinis kubas; kvarternioninė-Bézier formulė