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Dirichlet problem in one-dimensional billiard space with velocity dependent right-hand side

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Abstract. The paper brings multiplicity results for a Dirichlet problem in one-dimensional billiard space with right-hand side depending on the velocity of the ball, i.e. a problem in the form

$$x'' = f(t, x, x') \quad \text{if } x(t) \in \operatorname{int} K, \qquad x'(t+) = -x'(t-) \quad \text{if } x(t) \in \partial K,$$
$$x(0) = A, \quad x(T) = B,$$

where $T > 0$, $K = [0, R]$, $R > 0$, f is a Carathéodory function on $[0, T] \times K \times \mathbb{R}$, $A, B \in \operatorname{int} K$. Sufficient conditions ensuring the existence of at least two solutions having prescribed number of impacts with the boundary of the billiard table K are obtained. In particular, if the right-hand side has at most sublinear growth in the last variable, there exist infinitely many solutions of the problem.

Keywords: billiard problem; Dirichlet problem; multiplicity result; sublinear growth; linear growth

AMS Subject Classification: 34A37, 34B37

1 Introduction

Systems with impacts have been studied for a long time. One of the most known area of research is a billiard problem, where the paths of a moving ball inside of the "billiard table" are investigated. This type of problems was thoroughly studied (e.g. [10]) for various shapes of "tables" in various dimensions, but especially for a ball moving in a linear uniform way with absolutely elastic bounces at the boundary of the table. Quite recently, in [5] there were studied Dirichlet problems in billiard spaces, where the motion of the ball is not uniform anymore. Instead, the motion is determined by an impulsive second order ODE. The research continued in a similar direction in [16, 6, 7, 14]. Let us also mention the paper [9], where the author studied periodic solutions of this kind of problem. However, to the authors' knowledge, nothing is known for the case when the right-hand side depends on the derivative of a solution. The main purpose of this paper is to fill in the gap.

As it was mentioned, here, billiard problems are formulated and investigated in the framework of *impulsive differential equations* (IODEs). This theory enables to consider models driven by ODEs in which at certain instants the abrupt changes can take place. The changes are modelled as discontinuous – we call them impulses. The theory of IODEs is quite well developed (see [2, 11, 15]). The impulsive problems can be classified according to the times these impulses can occur: *fixed-times* vs. *state-dependent* problems. In the first class of problems, the instants at which the impulses occur are a priori known. This simplifies the investigation and many methods known for classical ODEs have been simply generalized for impulsive equations. There is a vast number of papers dealing with these problems (see e.g. [8, 12]). On the other hand, state-dependent problems are more challenging. The impulse instants depend on the state of the system, which brings a number of complications. The literature from this class is not so rich (see e.g. [1, 13, 3, 4]). The present paper falls into the second class. The common idea for investigation is to rewrite the impulsive problem into some auxiliary problem *without* impulses – this paper is no exception.

Here we deal with billiard problem type, which can be modeled as a boundary value problem for differential equation with state-dependent impulses

$$x'' = f(t, x, x'), \quad \text{for a.e. } t \in [0, T], \quad \text{if } x(t) \in (0, R), \tag{1}$$

$$x'(t+) = -x'(t-), \quad \text{if } x(t) \in \{0, R\}, \tag{2}$$

$$x(0) = A, \quad x(T) = B, \tag{3}$$

where $f \in \mathrm{Car}([0, T] \times [0, R] \times \mathbb{R})$, $T, R > 0$, $A, B \in (0, R)$.

Definition 1. The function $x \in \mathrm{C}([0, T])$ is called *a solution of problem* (1)*,* (2) if and only if

- $x([0, T]) \subset [0, R]$,
- for each interval $J \subset [0, T]$ for which $x(J) \subset (0, R)$, it is valid $x \in \mathrm{AC}^1(J)$ and x satisfies differential equation (1) a.e. on J,
- if $x(t) \in \{0, R\}$, then equality in (2) holds.

The number $p = \#\{t \in (0, T) : x(t) \in \{0, R\}\}$ is called *a number of impacts of the solution x with the boundary.* Moreover, if x also satisfies (3), we call it *a solution of problem* (1)–(3).

The main result of this paper is as follows:

Theorem 1. *Let $A, B \in (0, R)$, $f \in \mathrm{Car}([0, T] \times [0, R] \times \mathbb{R})$ and*

$$\begin{cases} \text{there exist } m \in \mathrm{L}^1([0, T]) \text{ and nonnegative, increasing } \varphi \in \mathrm{C}([0, \infty)) \text{ such that} \\ \big|f(t, x, y)\big| \leqslant m(t) + \varphi\big(|y|\big), \quad \text{for a.e. } t \in [0, T], \text{ all } x \in [0, R], \ y \in \mathbb{R}. \end{cases} \tag{4}$$

If there exist $p \in \mathbb{N}$ and $L > 0$ such that

$$\frac{T}{R}\big(\bar{m} + T\varphi(L)\big) + 1 \leqslant p \leqslant \frac{T}{R}\big(L - \bar{m} - T\varphi(L)\big) - 1, \tag{5}$$

where $\bar{m} = \int_0^T m(t)\,\mathrm{d}t$, then there exist at least two solutions of (1)–(3) *having exactly p impacts with the boundary.*

The paper is organized as follows. In Section 2 auxiliary problems are considered, main tools are built. Section 3 contains main results of the paper. Several examples showing usability of the results are given in the last section.

We use the following notation: Let $J \subset \mathbb{R}$ be an interval. Then, $\mathrm{L}^1(J)$ stands for the linear space of Lebesgue integrable functions on J; $\mathrm{C}(J)$ stands for the linear space of continous functions on J (esp. if J is compact, this space is equipped with the maximum norm $\|x\|_\infty = \max_{t\in J}|x(t)|$ forming a Banach space); $\mathrm{C}^1(J)$ is the linear space of functions having continuous derivatives on J and $\mathrm{AC}^1(J)$ is the linear space of functions having absolutely continuous derivatives on J. For $T > 0$ and $\mathcal{B} \subset \mathbb{R}^2$, by $\mathrm{Car}([0,T]\times\mathcal{B})$, we denote the set of all functions $f : [0,T]\times\mathcal{B} \to \mathbb{R}$ satisfying the Carathéodory conditions, i.e.

- $f(\cdot, x, y)$ is measurable on $[0,T]$ for each $(x,y) \in \mathcal{B}$,
- $f(t,\cdot,\cdot)$ is continuous on $\mathcal{B}$ for a.e. $t \in [0,T]$,
- for every compact set $K \subset \mathcal{B}$ there exists $m_K \in \mathrm{L}^1([0,T])$ such that

  $$\big|f(t,x,y)\big| \leqslant m_K(t)$$

  for a.e. $t \in [0,T]$ and every $(x,y) \in K$.

2 Auxiliary problems

In the whole section we consider the function $f \in \mathrm{Car}([0,T]\times[0,R]\times\mathbb{R})$ satisfying the condition (4).

We construct several auxiliary problems. First, we rewrite the impulsive differential equation (1), (2) into non-impulsive one (7). Since the right-hand side of this equation is not bounded in the last variable, we construct another auxiliary equation (10). And since the right-hand side of the last equation is not necessarily continuous in the second variable, we construct a sequence of differential equations (15). We are able to obtain solutions of boundary value problems for equations (15). At the end of this section we are able to get solutions of boundary value problems for equation (10) – see Lemma 4. This is utilized in the next section containing the proofs of the main results.

First, let us define functions

$$\triangle(x) = \begin{cases} x - 2kR, & \text{if } x \in [2kR, (2k+1)R), \\ 2(k+1)R - x, & \text{if } x \in [(2k+1)R, 2(k+1)R),\ k \in \mathbb{Z} \end{cases}$$

and

$$f^*(t,x,y) = \begin{cases} f(t, x-2kR, y), & \text{if } x \in (2kR, (2k+1)R), \\ -f\big(t, 2(k+1)R - x, -y\big), & \text{if } x \in \big((2k+1)R, 2(k+1)R\big),\ k \in \mathbb{Z}, \\ 0 & \text{else.} \end{cases} \tag{6}$$

We consider the following differential equation

$$y'' = f^*(t, y, y'). \tag{7}$$

By a solution of (7), we mean a function $y \in \mathrm{AC}^1([0,T])$ satisfying (7) a.e. on $[0,T]$.

Lemma 1. *Let y be a strictly monotone solution of (7), $y(0)/R, y(T)/R \notin \mathbb{Z}$. Then $x = \triangle \circ y$ is a solution of (1), (2) and has exactly*

$$\left| \left\lfloor \frac{y(0)}{R} \right\rfloor - \left\lfloor \frac{y(T)}{R} \right\rfloor \right|$$

impacts with the boundary.

Proof. Let y be a strictly increasing solution of (7) (the case for strictly decreasing solution is made similarly). We define

$$x(t) = \triangle(y(t)) = \begin{cases} y(t) - 2kR, & \text{if } y(t) \in \big[2kR, (2k+1)R\big), \\ 2(k+1)R - y(t), & \text{if } y(t) \in \big[(2k+1)R, 2(k+1)R\big),\ k \in \mathbb{Z}. \end{cases} \tag{8}$$

From (8) we see that x is continuous on $[0,T]$, its values lie in the interval $[0,R]$. Let us denote

$$m := \left\lfloor \frac{y(0)}{R} \right\rfloor, \qquad n := \left\lfloor \frac{y(T)}{R} \right\rfloor, \qquad p := \left| \left\lfloor \frac{y(0)}{R} \right\rfloor - \left\lfloor \frac{y(T)}{R} \right\rfloor \right|.$$

Then $p = |m-n| = n-m$, $mR < y(0) < (m+1)R$ and $nR < y(T) < (n+1)R$. From the continuity and monotonicity of y, it follows that there exist $0 < t_1 < t_2 < \cdots < t_p < T$ such that

$$y(t_i) = (m+i)R, \quad i = 1, 2, \dots, p.$$

Therefore $x(t) \in \{0, R\}$ iff $t = t_i, i \in \{1, 2, \dots, p\}$.

Let us put $t_0 = 0$, $t_{p+1} = T$. For $i = 0, 1, \dots, p$, $t \in (t_i, t_{i+1})$ we have $y(t) \in ((m+i)R, (m+i+1)R)$ and

$$x'(t) = (-1)^{m+i} y'(t), \quad t \in (t_i, t_{i+1})$$

i.e. x' is absolutely continuous on (t_i, t_{i+1}). Moreover

$$x'(t_i+) = \lim_{t \to t_i+} x'(t) = \lim_{t \to t_i+} (-1)^{m+i} y'(t) = (-1)^{m+i} y'(t_i+), \quad i = 0, 1, \dots, p$$

and

$$x'(t_i-) = \lim_{t \to t_i-} x'(t) = \lim_{t \to t_i-} (-1)^{m+i-1} y'(t) = (-1)^{m+i-1} y'(t_i-), \quad i = 1, 2, \dots, p+1.$$

Therefore x' is absolutely continuous on $[t_i, t_{i+1}]$, $i = 0, 1, \dots, p$ and x satisfy (2).

Now let $i \in \{0, 1, \dots, p\}$. First, let $k \in \mathbb{Z}$ be such that $y(t) \in (2kR, (2k+1)R)$ for each $t \in (t_i, t_{i+1})$. From (8) it follows that $x'(t) = y'(t)$ so together with (6), (7) and (8) we get

$$\begin{aligned} x''(t) &= \big(y(t) - 2kR\big)'' = y''(t) = f^*\big(t, y(t), y'(t)\big) \\ &= f\big(t, y(t) - 2kR, y'(t)\big) = f\big(t, x(t), x'(t)\big) \end{aligned}$$

for a.e. $t \in (t_i, t_{i+1})$. Similarly, let $k \in \mathbb{Z}$ be such that $y(t) \in ((2k+1)R, 2(k+1)R)$ for each $t \in (t_i, t_{i+1})$. From (8) we have $x'(t) = -y'(t)$ so we get

$$\begin{aligned} x''(t) &= \big(2(k+1)R - y(t)\big)'' = -y''(t) = -f^*\big(t, y(t), y'(t)\big) \\ &= -\big(- f\big(t, 2(k+1)R - y(t), -y'(t)\big)\big) = f\big(t, x(t), x'(t)\big) \end{aligned}$$

for a.e. $t \in (t_i, t_{i+1})$. Therefore x satisfies (1).

For each $L > 0$ we define another auxiliary function

$$f_L^*(t, x, y) = f^*\big(t, x, \max\big\{ - L, \min\{y, L\}\big\}\big), \quad \text{a.e. } t \in [0, T], \text{ all } x, y \in \mathbb{R} \tag{9}$$

and a corresponding differential equation

$$y'' = f_L^*(t, y, y'). \tag{10}$$

Again, by a solution of (10), we mean a function $y \in \mathrm{AC}^1([0, T])$ satisfying (10) a.e. on $[0, T]$.

Remark 1. It is easy to see that if y is a solution of equation (10) and $|y'(t)| \leqslant L$ for each $t \in [0, T]$, then y is also a solution of equation (7). The purpose of the function f_L^* lies in its boundedness according to the last variable – in particular

$$\big|f_L^*(t, x, y)\big| \leqslant m(t) + \varphi(L), \quad \text{for a.e. } t \in [0, T], \ x, y \in \mathbb{R}, \tag{11}$$

where m and φ are the functions from assumption (4).

In view of Remark 1 and Lemma 1 we see that in order to prove the existence of a solution to the problem (1)–(3), it is enough to find solutions y of the equation (10) satisfying

$$0 < y'(t) \leqslant L, \quad \forall t \in [0, T] \quad \text{or} \quad - L \leqslant y'(t) < 0, \quad \forall t \in [0, T]$$

and boundary conditions (3). The main obstacle in obtaining solutions of problem (10) lies in the fact that f_L^* is not a Carathéodory function, because the second condition may not apply. One of the possible methods[2] is to define sequence of auxiliary equations with Carathéodory right-hand sides.

Let us consider

$$\eta_n(x) = \begin{cases} \frac{2n}{R}(x - kR), & \text{if } x - kR \in \big[0, \frac{R}{2n}\big), \\ 1, & \text{if } x - kR \in \big(\frac{R}{2n}, R\big(1 - \frac{1}{2n}\big)\big), \\ \frac{2n}{R}((k+1)R - x), & \text{if } x - kR \in \big(R\big(1 - \frac{1}{2n}\big), R\big), \ k \in \mathbb{Z}, \end{cases} \tag{12}$$

[2] Another possibility is to consider Fillipov or Krasovskii regularization – see e.g. [14].

and

$$f^*_{L,n}(t,x,y)=\eta_n(x)f^*_L(t,x,y)\quad \text{a.e. } t\in[0,T],\ \text{all } x,y\in\mathbb{R}, \tag{13}$$

for $n\in\mathbb{N}$.

Remark 2. The functions η_n and $f^*_{L,n}$ have the following properties:

- η_n is R-periodic, continuous and piecewise linear on $\mathbb{R}$, $\eta_n(kR)=0$ for each $k\in\mathbb{Z}$, $0\leqslant\eta_n(x)\leqslant 1$ for each $x\in\mathbb{R}$,
- $\lim_{n\to\infty}\eta_n(x)=1$ for each $x\neq kR$, $k\in\mathbb{Z}$,
- $\lim_{n\to\infty}f^*_{L,n}(t,x,y)=f^*_L(t,x,y)$ for a.e. $t\in[0,T]$, all $(x,y)\in\mathbb{R}^2$, $x\neq kR$, $k\in\mathbb{Z}$,
- from Remark 1 it follows

$$\left|f^*_{L,n}(t,x,y)\right|\leqslant m(t)+\varphi(L)\quad \text{for a.e. } t\in[0,T],\ x,y\in\mathbb{R}, \tag{14}$$

- $f^*_{L,n}\in \mathrm{Car}([0,T]\times\mathbb{R}^2)$.

The last property of $f^*_{L,n}$ follows from the fact, that the possible discontinuity at $x=kR$ of f^*_L is cancelled out by multiplying it with the continuous function η_n vanishing at $x=kR$, $k\in\mathbb{Z}$.

Finally, we can define a sequence of (regular) auxiliary differential equations

$$y''=f^*_{L,n}(t,y,y'). \tag{15}$$

By its solution, we mean a function $y\in\mathrm{AC}^1([0,T])$ satisfying it a.e. on $[0,T]$.

Let us solve the formulated problems. First, we find solutions for boundary value problems for equations (15).

Lemma 2. *Let $L>0$, $A,B\in\mathbb{R}$, $A/R,B/R\notin\mathbb{Z}$. Then for each $n\in\mathbb{N}$ there exists at least one solution y_n of boundary value problem* (15), (3) *satisfying*

$$\left\|y_n'-\frac{B-A}{T}\right\|_\infty\leqslant\bar{m}+T\varphi(L) \tag{16}$$

and

$$\|y_n\|_\infty\leqslant K,\qquad \|y_n'\|_\infty\leqslant K_1, \tag{17}$$

where constants K and K_1 are independent of n.

Proof. Let us consider

$$G(t,s)=\begin{cases}\frac{t(s-T)}{T}, & \text{if } 0\leqslant t\leqslant s\leqslant T,\\ \frac{s(t-T)}{T}, & \text{if } 0\leqslant s<t\leqslant T,\end{cases}$$

which is the Green's function of the boundary value problem $y''=f(t)$, $y(0)=0$, $y(T)=0$. Let us define an operator $\mathcal{F}_n:\mathrm{C}^1([0,T])\to\mathrm{C}^1([0,T])$ by

$$\mathcal{F}_ny(t)=\frac{t}{T}B+\frac{T-t}{T}A+\int_0^T G(t,s)f^*_{L,n}(s,y(s),y'(s))\,\mathrm{d}s$$

for $y \in \mathrm{C}^1([0,T])$, $t \in [0,T]$. We also define the set

$$\Omega = \bigl\{ y \in \mathrm{C}^1\bigl([0,T]\bigr) : \|y\|_\infty \leqslant K,\ \|y'\|_\infty \leqslant K_1 \bigr\},$$

where we put

$$K = |B| + |A| + \frac{T}{4}\bigl(\bar{m} + T\varphi(L)\bigr), \qquad K_1 = \frac{|B-A|}{T} + \bar{m} + T\varphi(L).$$

Using Remark 2, it is standard to show that $\mathcal{F}_n$ is a completely continuous operator. Moreover, for $y \in \mathrm{C}^1([0,T])$, $t \in [0,T]$ we have

$$\begin{aligned}
\bigl|(\mathcal{F}_n y)(t)\bigr| &\leqslant |B| + |A| + \int_0^T \bigl|G(t,s)\bigr| \cdot \bigl|f^*_{L,n}(s, y(s), y'(s))\bigr| \,\mathrm{d}s \\
&\leqslant |B| + |A| + \operatorname{ess\,sup}_{[0,T]^2} |G| \int_0^T \bigl(m(s) + \varphi(L)\bigr)\,\mathrm{d}s = K
\end{aligned}$$

and

$$\begin{aligned}
\bigl|(\mathcal{F}_n y)'(t)\bigr| &= \left| \frac{B-A}{T} + \int_0^T \frac{\partial G}{\partial t}(t,s) f^*_{L,n}\bigl(s, y(s), y'(s)\bigr)\,\mathrm{d}s \right| \\
&\leqslant \frac{|B-A|}{T} + \operatorname{ess\,sup}_{[0,T]^2} \left|\frac{\partial G}{\partial t}\right| \int_0^T \bigl(m(s) + \varphi(L)\bigr)\,\mathrm{d}s = K_1,
\end{aligned}$$

i.e. $\mathcal{F}_n(\Omega) \in \Omega$. According to Schauder Fixed Point Theorem there exists at least one fixed point $y_n \in \Omega$ of the operator $\mathcal{F}_n$, i.e. it is also a solution of (15), (3). Moreover

$$\begin{aligned}
\left| y_n'(t) - \frac{B-A}{T} \right| &= \left| (\mathcal{F}_n y_n)'(t) - \frac{B-A}{T} \right| \\
&= \left| \int_0^T \frac{\partial G}{\partial t}(t,s) f^*_{L,n}\bigl(s, y_n(s), y_n'(s)\bigr)\,\mathrm{d}s \right| \leqslant \bar{m} + T\varphi(L)
\end{aligned}$$

for each $t \in [0,T]$. □

The sequence of solutions from Lemma 2 generates a solution of certain boundary value problems for equation (10).

Lemma 3. *Let $L > 0$, $A, B \in \mathbb{R}$, $A/R, B/R \notin \mathbb{Z}$ satisfy*

$$\bar{m} + T\varphi(L) < \frac{|B-A|}{T}. \tag{18}$$

Then there exists at least one solution y of boundary value problem (10), (3) *such that*

$$\left\| y' - \frac{B-A}{T} \right\|_\infty \leqslant \bar{m} + T\varphi(L), \tag{19}$$

and $y'(t) \neq 0$ for every $t \in [0,T]$.

Proof. According to Lemma 2, for every $n \in \mathbb{N}$ there exists at least one solution y_n satisfying (16) and (17). From (17) it follows that $\{y_n\}$, $\{y_n'\}$ are equibounded sequences of continuous functions. Moreover for all $n \in \mathbb{N}$ and a.e. $t \in [0, T]$ we have

$$\left|y_n''(t)\right| = \left|f_{L,n}^*\big(t, y_n(t), y_n'(t)\big)\right| \leqslant m(t) + T\varphi(L),$$

so it follows that $\{y_n'\}$ and $\{y_n\}$ are equicontinuous sequences. According to Arzelà–Ascoli Theorem there exist $y \in \mathrm{C}^1([0,T])$ and a subsequence $\{y_{k_n}\}$ such that $y_{k_n} \to y$ in $\mathrm{C}^1([0,T])$. From (16) we obtain (19), i.e.

$$\frac{B-A}{T} - \bar{m} - T\varphi(L) \leqslant y'(t) \leqslant \frac{B-A}{T} + \bar{m} + T\varphi(L), \quad \forall t \in [0,T].$$

If $B - A > 0$, then from (18) it follows that

$$\frac{B-A}{T} - \bar{m} - T\varphi(L) > 0$$

and therefore $y'(t) > 0$ for all $t \in [0,T]$.

If $B - A < 0$, then from (18) it follows that

$$0 > \frac{B-A}{T} + \bar{m} + T\varphi(L)$$

and therefore $y'(t) < 0$ for all $t \in [0,T]$. In both cases, y is strictly monotone. Then there exists a finite set $T_y = \{t_1, \dots, t_p\} \subset (0,T)$, $t_1 < t_2 < \dots < t_p$ such that for all $t \in [0,T]$

$$\frac{y(t)}{R} \in \mathbb{Z} \quad \Leftrightarrow \quad t \in T_y.$$

Let us put $t_0 = 0$, $t_{p+1} = T$. Consider $i \in \{0, \dots, p\}$. Then there exists $k \in \mathbb{Z}$ such that $y(t) \in (kR, (k+1)R)$ for all $t \in (t_i, t_{i+1})$. For each $s_1, s_2 \in (t_i, t_{i+1})$, $s_1 < s_2$ we have

$$\begin{aligned} y'(s_2) - y'(s_1) &= \lim_{n\to\infty} \big(y_{k_n}'(s_2) - y_{k_n}'(s_1)\big) = \lim_{n\to\infty} \int_{s_1}^{s_2} y_{k_n}''(s)\,\mathrm{d}s \\ &= \lim_{n\to\infty} \int_{s_1}^{s_2} f_{L,k_n}^*\big(s, y_{k_n}(s), y_{k_n}'(s)\big)\,\mathrm{d}s. \end{aligned}$$

Since the convergence of sequence $\{y_{k_n}\}$ is uniform and y is strictly monotone on (t_i, t_{i+1}) then

$$y_{k_n}(t) - kR \in \left(\frac{R}{2k_n}, R\left(1 - \frac{1}{2k_n}\right)\right), \quad \forall t \in [s_1, s_2]$$

for sufficiently large n, i.e.

$$f_{L,k_n}^*\big(s, y_{k_n}(s), y_{k_n}'(s)\big) = f_L^*\big(s, y_{k_n}(s), y_{k_n}'(s)\big) \quad \text{for a.e. } s \in [s_1, s_2]$$

for sufficiently large n. Then

$$\begin{aligned} \lim_{n\to\infty} \int_{s_1}^{s_2} f_{L,k_n}^*\big(s, y_{k_n}(s), y_{k_n}'(s)\big)\,\mathrm{d}s &= \lim_{n\to\infty} \int_{s_1}^{s_2} f_L^*\big(s, y_{k_n}(s), y_{k_n}'(s)\big)\,\mathrm{d}s \\ &= \lim_{n\to\infty} \int_{s_1}^{s_2} f_L^*\big(s, y(s), y'(s)\big)\,\mathrm{d}s. \end{aligned}$$

The last equality follows from the fact that $\{y_{k_n}\}$, $\{y'_{k_n}\}$ are uniformly convergent and

$$f_L^* \in \operatorname{Car}\bigl([0,T] \times \bigl(kR,(k+1)R\bigr) \times \mathbb{R}\bigr).$$

Therefore

$$y'(s_2) - y'(s_1) = \int_{s_1}^{s_2} f_L^*(s, y(s), y'(s))\,\mathrm{d}s$$

for each $s_1, s_2 \in (t_i, t_{i+1})$, $s_1 < s_2$, i.e. y satisfies ODE (10) a.e. on (t_i, t_{i+1}). $\square$

Finally, we obtain solutions of boundary value problem for the auxiliary equation (7).

Lemma 4. *Let $L > 0$, $A, B \in \mathbb{R}$, $A/R, B/R \notin \mathbb{Z}$ satisfy*

$$\bar{m} + T\varphi(L) < \frac{|B-A|}{T} \leqslant L - \bar{m} - T\varphi(L). \tag{20}$$

Then there exists at least one solution y of the boundary value problem (7), (3) *satisfying*

$$0 < \bigl|y'(t)\bigr| \leqslant L \quad t \in [0,T].$$

Proof. From the first equality in (20) we see that we can use Lemma 3, implying the existence of a solution y_L of problem (10), (3) having nonzero derivative. It is enough to prove that $\|y'_L\|_\infty \leqslant L$. From (19) we have

$$\frac{B-A}{T} - \bar{m} - T\varphi(L) \leqslant y'_L(t) \leqslant \frac{B-A}{T} + \bar{m} + T\varphi(L). \tag{21}$$

If $B - A > 0$, from (20) we see

$$0 < \frac{B-A}{T} - \bar{m} - T\varphi(L) \quad \text{and} \quad \frac{B-A}{T} + \bar{m} + T\varphi(L) \leqslant L$$

and therefore by (21) we have

$$0 < y'_L(t) \leqslant L, \quad \text{for each } t \in [0,T].$$

If $B - A < 0$, from (20) we see

$$\bar{m} + T\varphi(L) < \frac{A-B}{T} \leqslant L - \bar{m} - T\varphi(L),$$

i.e.

$$\frac{B-A}{T} + \bar{m} + T\varphi(L) < 0 \quad \text{and} \quad -L \leqslant \frac{B-A}{T} - \bar{m} - T\varphi(L),$$

and therefore by (21) we have

$$-L \leqslant y'_L(t) < 0, \quad \text{for each } t \in [0,T].$$

In both cases $|y'(t)| \leqslant L$ for $t \in [0,T]$. $\square$

3 Main results

Now, we are ready to prove the main result of this paper which is Theorem 1.

Proof. (of Theorem 1) Let p and L satisfy (5). We consider two boundary conditions

$$y(0) = A, \quad y(T) = B_i \tag{22}$$

for $i = p$ and $i = -p$, where

$$B_i = \begin{cases} iR + B, & \text{if } i \text{ is even,} \\ (i+1)R - B, & \text{if } i \text{ is odd.} \end{cases}$$

Since $iR < B_i < (i+1)R$ and $-R < -A < 0$ it follows $(i-1)R < B_i - A < (i+1)R$. For both $i = \pm p$ we have

$$\frac{(p-1)R}{T} < \frac{|B_i - A|}{T} < \frac{(p+1)R}{T}.$$

From (5) it follows that

$$\frac{(p-1)R}{T} \geqslant \frac{T}{R}\big(\bar{m} + T\varphi(L)\big)\frac{R}{T} = \bar{m} + T\varphi(L)$$

and

$$\frac{(p+1)R}{T} \leqslant \frac{T}{R}\big(L - \bar{m} - T\varphi(L)\big)\frac{R}{T} = L - \bar{m} - T\varphi(L).$$

Therefore the condition (20) is satisfied for boundary value problem (7), (3) with $B = B_{\pm p}$. From Lemma 4 it follows that there exist strictly increasing solution y_1 of (7), (22) with $i = p$ and strictly decreasing solution y_2 of (7), (22) with $i = -p$. From Lemma 1 it follows that

$$x_1 = \triangle \circ y_1, \quad x_2 = \triangle \circ y_2$$

are two (distinct) solutions of (1), (2) satisfying

$$x_{1,2}(0) = \triangle(y_{1,2}(0)) = A, \quad x_{1,2}(T) = \triangle(y_{1,2}(T)) = \triangle(B_{\pm p}) = B,$$

i.e. the boundary condition (3), having exactly

$$\left|\left\lfloor \frac{y_{1,2}(0)}{R} \right\rfloor - \left\lfloor \frac{y_{1,2}(T)}{R} \right\rfloor\right| = \left|\left\lfloor \frac{A}{R} \right\rfloor - \left\lfloor \frac{B_{\pm p}}{R} \right\rfloor\right| = |0 - (\pm p)| = p$$

impacts with the boundary. □

Corollary 1. *Let* $A, B \in (0, R)$*,* $f \in \mathrm{Car}([0,T] \times [0,R] \times \mathbb{R})$*. If there exists* $m \in \mathrm{L}^1([0,T])$ *such that*

$$\big|f(t,x,y)\big| \leqslant m(t), \quad \textit{for a.e. } t \in [0,T], \ \textit{all } x \in [0,R], \ y \in \mathbb{R}, \tag{23}$$

then for each $p \in \mathbb{N}$ *satisfying*

$$p \geqslant \frac{T}{R}\bar{m} + 1, \tag{24}$$

where $\bar{m} = \int_0^T m(t)\,\mathrm{d}t$*, there exist at least two solutions of* (1)–(3) *having exactly* p *impacts with the boundary.*

Proof. Let p satisfy (24). From (23) it follows that f satisfies (4) with $\varphi \equiv 0$. We take $L > 0$ such that

$$p \leqslant \frac{T}{R}(L - \bar{m}) - 1.$$

Then (5) is satisfied and theorefore (1)–(3) has at least two solutions having exactly p impacts. □

Corollary 1 can be understood as a simple generalization of [16, Theorem 2], where the right-hand side of the differential equation depends on the variable y' now.

Let us present some more efficient sufficient conditions than those of Theorem 1.

Theorem 2. *Let $A, B \in (0, R)$, $f \in \mathrm{Car}([0, T] \times [0, R] \times \mathbb{R})$ satisfy (4) and*

$$\limsup_{y \to \infty} \frac{\varphi(y)}{y} < \frac{1}{2T}.$$

Then (1)–(3) has infinitely many solutions. In particular, there exists an increasing sequence of positive integers $\{p_n\}$ such that for each $n \in \mathbb{N}$ there exists at least two solutions of (1)–(3) with exactly p_n impacts with the boundary.

Proof. Let us denote

$$\psi_1(L) = \frac{T}{R}\big(\bar{m} + T\varphi(L)\big) + 1, \qquad \psi_2(L) = \frac{T}{R}\big(L - \bar{m} - T\varphi(L)\big) - 1, \quad L > 0.$$

The condition (5) is therefore equivalent to

$$\psi_1(L) \leqslant p \leqslant \psi_2(L)$$

for $p \in \mathbb{N}$, $L \in (0, \infty)$. We have

$$\begin{aligned}\lim_{L \to \infty} \psi_2(L) - \psi_1(L) &= \lim_{L \to \infty} \frac{T}{R}\big(L - 2\bar{m} - 2T\varphi(L)\big) - 2 \\ &= \lim_{L \to \infty} L \cdot \left(\frac{T}{R}\left(1 - \frac{2\bar{m}}{L} - 2T\frac{\varphi(L)}{L}\right) - \frac{2}{L}\right) = \infty.\end{aligned}$$

Therefore for each positive integer n there exists $L_n > 0$ such that

$$\psi_2(L_n) - \psi_1(L_n) > n,$$

i.e. there exist (at least) n positive integers $p_1, \dots, p_n$ such that $p_i \in [\psi_1(L_n), \psi_2(L_n)]$ for $i = 1, \dots, n$. According to Theorem 1, for each $i = 1, \dots, n$ there exist at least two solutions of (1)–(3) having exactly p_i impacts with the boundary. Since n can be arbitrary there exist infinitely many solutions of (1)–(3). □

4 Examples

Let us present some applications of our results.

Example 1. Let us consider boundary value problem (1)–(3), where

$$f(t,x,y) = t^\alpha + \lambda |x|^\beta \operatorname{sgn} x + \omega \sin y,$$

where $\alpha > -1$, $\beta > 0$, $\lambda, \omega \in \mathbb{R}$. Since

$$\bigl|f(t,x,y)\bigr| \leqslant t^\alpha + |\lambda| R^\beta + |\omega| =: m(t) \in \mathrm{L}^1\bigl([0,T]\bigr).$$

The assumptions of Corollary 1 are satisfied and therefore for each $p \geqslant T\bar{m}/R + 1$ there exist at least two solutions having exactly p impacts, where

$$\bar{m} = \int_0^T t^\alpha + |\lambda| R + |\omega| \,\mathrm{d}t = \left(\frac{T^\alpha}{\alpha+1} + |\lambda| R + |\omega|\right) T.$$

Example 2. Let us consider boundary value problem (1)–(3), where

$$f(t,x,y) = t^\alpha + \lambda |x|^\beta \operatorname{sgn} x + \omega |y|^\gamma \operatorname{sgn} y,$$

where $\alpha > -1, \beta > 0$, $\gamma \in (0,1)$. The assumptions of Theorem 2 are satisfied for

$$m(t) = t^\alpha + \lambda R^\beta, \qquad \varphi(y) = |\omega| \cdot y^\gamma.$$

Since

$$\lim_{y\to\infty} \frac{\varphi(y)}{y} = \lim_{y\to\infty} |\omega| y^{\gamma-1} = 0 < \frac{1}{2T},$$

the problem (1)–(3) has infinitely many solutions with arbitrary large number of impacts with impulses. Using Theorem 1 we could get more detailed information. Let us define

$$\psi_1(L) = \frac{T}{R}\bigl(\bar{m} + T\varphi(L)\bigr) + 1 = \frac{T^2}{R}\left(\frac{T^\alpha}{\alpha+1} + |\lambda| R^\beta + |\omega| L^\gamma\right) + 1$$

and

$$\psi_2(L) = \frac{T}{R}\bigl(L - \bar{m} - T\varphi(L)\bigr) - 1 = \frac{T}{R}\left(L - T\left(\frac{T^\alpha}{\alpha+1} + |\lambda| R^\beta + |\omega| L^\gamma\right)\right) - 1$$

For the multicity result we need to find a couple $(L,p) \in (0,\infty) \times \mathbb{N}$ such that $\psi_1(L) < p < \psi_2(L)$. For the following values of the parameters

$$T = 1, \quad R = 1, \quad \alpha = 1, \quad \beta = 0.5, \quad \gamma = 0.5, \quad \omega = 0.5 \quad \text{and} \quad \lambda = 0.1$$

we can see the graphs of the functions ψ_1 and ψ_2 in Fig. 1. For such values, it can be easily seen that for each $p \geqslant 3$ there exist at least two solutions with at least p impacts.

Example 3. Let us consider boundary value problem (1)–(3) with

$$f(t,x,y) = t^\alpha + \lambda |x|^\beta \operatorname{sgn} x + \omega y,$$

where $\alpha > -1, \beta, \gamma > 0$, $\omega \in \mathbb{R}$ such that $2T|\omega| < 1$. The assumptions of Theorem 2 are satisfied for

$$m(t) = t^\alpha + \lambda R^\beta, \quad \varphi(y) = |\omega| y.$$

By the theorem the problem (1)–(3) has infinitely many solutions with arbitrary large number of impacts with the boundary.

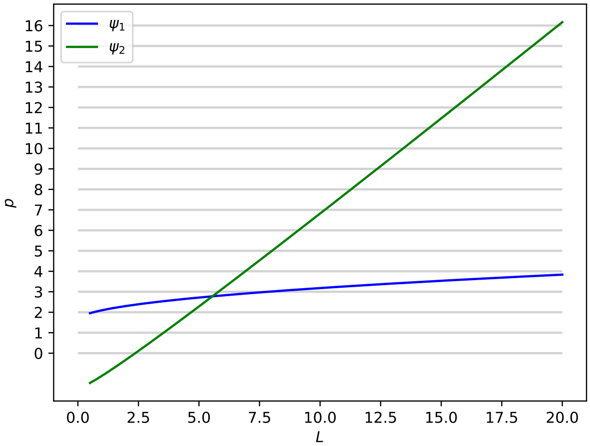

Fig. 1. Graphs of $\psi_{1,2}$ from Example 2.

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REZIUMĖ

Dirichlė uždavinys vienmatėje biliardo erdvėje su nuo greičio priklausoma dešine puse

V. Krajščáková, J. Tomeček

Straipsnyje pateikiami vienmačio Dirichlė uždavinio rezultatai biliardo erdvėje su dešine puse, priklausoma nuo kamuoliuko greičio, t. y. nagrinėjamas uždavinys

$$x'' = f(t, x, x') \quad \text{if } x(t) \in \operatorname{int} K, \qquad x'(t+) = -x'(t-) \quad \text{if } x(t) \in \partial K,$$
$$x(0) = A, \quad x(T) = B,$$

čia $T > 0$, $K = [0, R]$, $R > 0$, f yra Caratheodorio funkcija $[0, T] \times K \times \mathbb{R}$, $A, B \in \operatorname{int} K$. Gautos pakankamos sąlygos, užtikrinančios egzistavimą bent dviejų sprendinių, turinčių nustatytą skaičių smūgių su biliardo stalo K kraštu. Atskiru atveju, jei dešinėje pusėje pagal paskutinį kintamąjį turime subtiesinį augimą, egzistuoja be galo daug šio uždavinio sprendinių.

Raktiniai žodžiai: biliardo uždavinys; Dirichlė uždavinys; kartotinumo rezultatas; subtiesinis augimas; tiesinis augimas