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Evaluating a double integral using Euler's method and Richardson extrapolation

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Abstract. We transform a double integral into a second-order initial value problem, which we solve using Euler's method and Richardson extrapolation. For an example we consider, we achieve accuracy close to machine precision ($\sim 10^{-13}$). We find that the algorithm is capable of determining the error curve for an arbitrary cubature formula, and we use this feature to determine the error curve for a Simpson cubature rule. We also provide a generalization of the method to the case of nonlinear limits in the outer integral.

Keywords: cubature; double integral; Euler; Richardson extrapolation; error

AMS Subject Classification: 65D30, 65D32

Introduction

Many techniques for evaluating multiple integrals have been developed (see, for example, [24, 17, 8, 26, 21, 6, 3, 4, 5, 2, 13, 12, 7, 11, 10, 15, 9, 1, 14, 16, 22, 23, 25, 18, 20, 19, 27]). In this paper, we add to this pool of knowledge: we transform a double integral into a pure quadrature second-order initial value problem, which we solve using Euler's method and Richardson extrapolation (at the time of writing, we have not found evidence of this technique elsewhere). We will derive constraints on the integrand via Leibniz differentiation and Lipschitz continuity. We will show how the algorithm can be used to determine the error curve of an arbitrary cubature rule, and we will offer a generalization to the case of nonlinear limits in the outer integral.

Relevant Concepts

Here, we present concepts and notation relevant to the study.

- Let $f(x,t)$ be a real-valued function, i.e. $f:\mathbb{R}^2 \to \mathbb{R}$, that is Riemann integrable in both variables. We will assume that f is suitably smooth so that all relevant derivatives in our analysis exist, in particular $\partial f/\partial x$.

- Define

$$C(x) \equiv \int_{x_0}^{X} \int_{t_0(x)}^{t_1(x)} f(x,t)dtdx \tag{1}$$

  as the *double integral* of interest, noting that the limits of the inner integral may be functions of x. We necessarily require $t_0(x) > -\infty$, $t_1(x) < \infty$ for $x \in [x_0, X]$.

- Let M denote the exact numerical value of some mathematical object, such as an integral. Let $N(h)$ denote a numerical approximation to M that is dependent on an adjustable parameter h. Assume that

$$N(h) = M + K_1(x)h + K_2(x)h^2 + K_3(x)h^3 + K_4(x)h^4 + \dots \tag{2}$$

  In other words, the approximation error in $N(h)$ is a power series in h. *Richardson extrapolation* is a process by which values of $N(h)$, for differing values of h, can be combined linearly to yield approximations that are of higher order than the original approximation $N(h)$. Details of this procedure, relevant to the current paper, are provided in the Appendix.

- The well-known *Euler's method* is a first-order Runge-Kutta method for numerically solving an initial value problem, and it has an approximation error of the form in (2).

- The following thorem will be used later:

Theorem 1 [Leibniz]. *Let the function $f(x,t)$ be such that $f(x,t)$ and $\frac{\partial f(x,t)}{\partial x}$ are continuous in x and t in some region of the xt-plane, which includes $t_0(x) \leqslant t \leqslant t_1(x)$, $x_0 \leqslant x \leqslant X$. Additionally, assume that $t_0(x)$ and $t_1(x)$ and their first derivatives are continuous on $[x_0, X]$. Then, for $x \in [x_0, X]$,*

$$\frac{d}{dx}\left(\int\limits_{t_0(x)}^{t_1(x)} f(x,t)dtdx\right) = f(x,t_1(x))\frac{dt_1(x)}{dx} - f(x,t_0(x))\frac{dt_0(x)}{dx} + \int_{t_0(x)}^{t_1(x)} \frac{\partial f(x,t)}{\partial x}dtdx.$$

Theory

Assuming the conditions of Theorem 1 are satisfied, differentiating (1) with respect to x gives

$$C'(x) = \int_{t_0(x)}^{t_1(x)} f(x,t)dt,$$

$$C''(x) = f\big(x, t_1(x)\big)t_1'(x) - f\big(x, t_0(x)\big)t_0'(x) + \int_{t_0(x)}^{t_1(x)} \frac{\partial f(x,t)}{\partial x}dt$$

via the Leibniz rule (see Theorem 1). Clearly, this result implies the integrability of $\frac{\partial f(x,t)}{\partial x}$ with respect to t, on $[t_0(x), t_1(x)]$ for $x \in [x_0, X]$.

Hence, we have

$$C' = Z,$$

$$Z' = f(x, t_1(x))t_1'(x) - f(x, t_0(x))t_0'(x) + \int_{t_0(x)}^{t_1(x)} \frac{\partial f(x,t)}{\partial x}dt \equiv g(x) \tag{3}$$

with initial values

$$C(x_0) = \int_{x_0}^{x_0}\int_{t_0(x)}^{t_1(x)} f(x,t)dtdx = 0,$$

$$Z(x_0) = C'(x_0) = \int_{t_0(x_0)}^{t_1(x_0)} f(x_0,t)dt.$$

Of course, this is a second-order system, compactly expressed as

$$\begin{bmatrix} C' \\ Z' \end{bmatrix} = \begin{bmatrix} Z \\ g(x) \end{bmatrix}. \tag{4}$$

We define the nodes $\{x_0 < x_1 < x_2 < \cdots < x_i < \cdots < x_n < X\}$, where we treat the upper limit X as the final node in the set. The nodes $\{x_0, x_1, x_2, \ldots, x_i, \ldots, x_n\}$ are uniformly spaced; the spacing is denoted by the stepsize h. The node x_n is chosen such that $X - x_n \leqslant h$. Also, we define $C_i \equiv C(x_i), Z_i \equiv Z(x_i)$ and so on. Consequently, Euler's Method applied to this system can be written in the form

$$\begin{bmatrix} C_{i+1} \\ Z_{i+1} \end{bmatrix} = \begin{bmatrix} C_i \\ Z_i \end{bmatrix} + h \begin{bmatrix} Z_i \\ g(x_i) \end{bmatrix}.$$

Solving this system numerically yields the values $C(x_i)$ and, of course, $C(X)$.

If we demand Lipschitz continuity in x for the RHS of (4), we must have that the derivatives

$$\frac{\partial Z}{\partial x} = g(x),$$

$$\frac{\partial g}{\partial x} = \frac{\partial}{\partial x}\big(f(x,t_1(x))t_1'(x)\big) - \frac{\partial}{\partial x}\big(f(x,t_0(x))t_0'(x)\big) + \frac{\partial}{\partial x}\Bigg(\int\limits_{t_0(x)}^{t_1(x)} \frac{\partial f(x,t)}{\partial x}dt\Bigg)$$

$$= f_x\big(x,t_1(x)\big)t_1'(x) + f\big(x,t_1(x)\big)t_1''(x) - f_x(x,t_0(x))t_0'(x) - f(x,t_0(x))t_0''(x)$$

$$+ \frac{\partial f(x,t_1(x))}{\partial x}t_1'(x) - \frac{\partial f(x,t_0(x))}{\partial x}t_0'(x) + \int\limits_{t_0(x)}^{t_1(x)} \frac{\partial^2 f(x,t)}{\partial x^2}dt$$

are Lipschitz continuous on $[x_0, X]$. Also, we require the integrability of $\frac{\partial^2 f(x,t)}{\partial x^2}$ with respect to t, on $[t_0(x), t_1(x)]$ for $x \in [x_0, X]$. These conditions, together with those pertaining to Leibniz differentiation of (1), serve to characterise the problem (1).

Numerical Example

Consider

$$C(x) = \int_1^5 \int_{x/5}^{x^2+1} \sin(xt)dtdx \tag{5}$$

which gives, with $f(x,t) = \sin(xt)$,

$$Z' = 2x\sin(x^3 + x) - \frac{\sin(\frac{x^2}{5})}{5} + \int_{x/5}^{x^2+1} t\cos t dt. \tag{6}$$

The one-dimensional integral in (6) is evaluated using composite Gaussian quadrature [19] to a precision of 10^{-15}, although any suitably accurate technique can be used.

The initial values are

$$C(1) = 0,$$

$$Z(1) = \int_{1/5}^{2} \sin t dt = 1.396213414388384.$$

Clearly, we also have $x_0 = 1, X = 5, t_0(x) = x/5$ and $t_1(x) = x^2 + 1$.

Error control

It is possible to choose the stepsize h so as to achieve a desired level of accuracy in the computation of $C(x)$. Terminology and notation used in this section is described in the Appendix, and the reader is thereto referred.

With a moderately small value of h (we used $h = 0.01$ in our example), we use Euler's method to obtain solutions of (4), at the nodes $\{x_0 = 1, x_1, x_2, \ldots, x_i, \ldots,$

$X = 5\}$, for the various stepsizes required to construct $M_4(x,h)$ and $M_5(x,h)$ via Richardson extrapolation. We then perform the computations

$$\begin{aligned} M_4(x_i,h) - M_5(x_i,h) &= \widetilde{K}_4(x_i)h^4 - O(h^5) \\ &\approx \widetilde{K}_4(x_i)h^4 \\ \Rightarrow \widetilde{K}_4(x_i) &= \frac{M_4(x_i,h) - M_5(x_i,h)}{h^4} \end{aligned}$$

at each of the nodes $\{x_0 = 1, x_1, x_2, \dots, x_i, \dots, X = 5\}$. Hence, we have the error coefficient $\widetilde{K}_4$ at each node. We note that, due to the linear combination necessary to construct $M_4(x,h)$, $\widetilde{K}_4$ is not necessarily equal to K_4. In fact, we have $K_4 = 64\widetilde{K}_4$.

The sequence of calculations below then allows a new stepsize to be found, consistent with a user-defined tolerance ε.

$$\begin{aligned} h^* &= \left(\frac{\varepsilon}{\max_{x_i} |\widetilde{K}_4(x_i)|}\right)^{\frac{1}{4}}, \\ n &= \left\lceil \frac{x - x_0}{h^*} \right\rceil, \\ h &= \frac{x - x_0}{n}. \end{aligned}$$

The Euler/Richardson algorithm is then repeated using this new stepsize, ultimately leading to a new $M_4(x_i,h)$ and, in particular, $M_4(5,h) \approx C(5)$.

Results

Applying the algorithm to (5) gives the stepsizes needed for various tolerances, shown in Table 1.

Table 1. Tolerances and corresponding stepsizes for evaluating the double integral in (5).

ε	10^{-14}	10^{-12}	10^{-10}
h	2×10^{-4}	6.3×10^{-4}	2×10^{-3}
ε	10^{-8}	10^{-6}	10^{-4}
h	6.3×10^{-3}	2×10^{-2}	6.3×10^{-2}

In Fig. 1 we show $C(x)$ obtained from $M_4(x_i,h)$ with $h = 6.3 \times 10^{-4}$. In Fig. 2, we show the associated coefficients $K_1(x), K_2(x), K_3(x)$ and $K_4(x)$. In Fig. 3, we show error curves for $C(x)$ obtained from both $M_2(x_i,h)$ and $M_4(x_i,h)$, again with $h = 6.3 \times 10^{-4}$. Clearly, $M_4(x_i,h)$ appears to achieve a tolerance of $\sim 10^{-12}$ over the entire interval.

Of course, the original objective was to determine $C(5)$, and so

$$C(5) \approx M_4(5, 6.3 \times 10^{-4}) = 0.630635228375177$$

which is within 8.3×10^{-13} of the exact value.

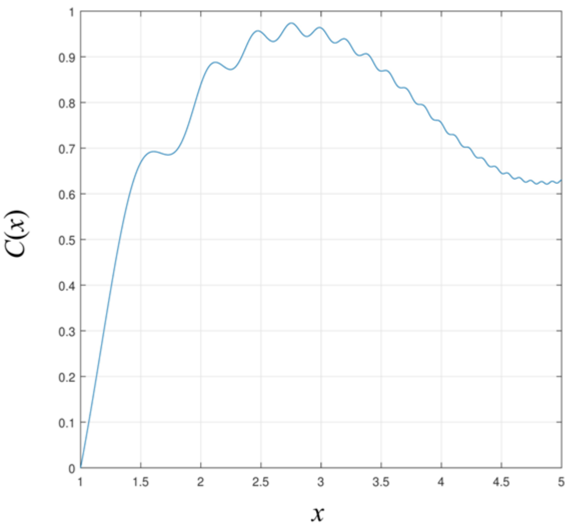

Fig. 1. $C(x)$ obtained from $M_4(x_i, h)$ with $h = 6.3 \times 10^{-4}$.

Possible applications

Apart from the obvious task of determining $C(x)$, we can also use this algorithm to find the error term for a given cubature rule $Q(x)$. If we include $Q(x)$ in (1), we have

$$\int_{x_0}^{X} \int_{t_0(x)}^{t_1(x)} f(x,t)dtdx = Q(x) + C(x)$$

which yields the system

$$C' = Z$$

$$Z' = f(x, t_1(x))t_1'(x) - f(x, t_0(x))t_0'(x) + \int_{t_0(x)}^{t_1(x)} \frac{\partial f(x,t)}{\partial x} dt - Q''(x) \equiv g(x),$$

with initial values

$$C(x_0) = \int_{x_0}^{x_0} \int_{t_0(x)}^{t_1(x)} f(x,t)dtdx - Q(x_0) = -Q(x_0),$$

$$Z(x_0) = C'(x_0) = \int_{t_0(x_0)}^{t_1(x_0)} f(x_0,t)dt - Q'(x_0).$$

In this case, $C(x)$ now plays the role of an *error term* or a *correction term* (earlier, we effectively had $Q(x) = 0$).

For example, applying Simpson's Rule to our example yields the cubature expression $Q(x)$ shown in the Appendix. Then, applying our algorithm with $h = 4 \times 10^{-3}$, we find the curves $Q(x)$ and $C(x)$ shown in Fig. 4. Comparing $Q(x)$ to the curve in Fig. 1, we see that the cubature rule is very inaccurate. However, when the correction

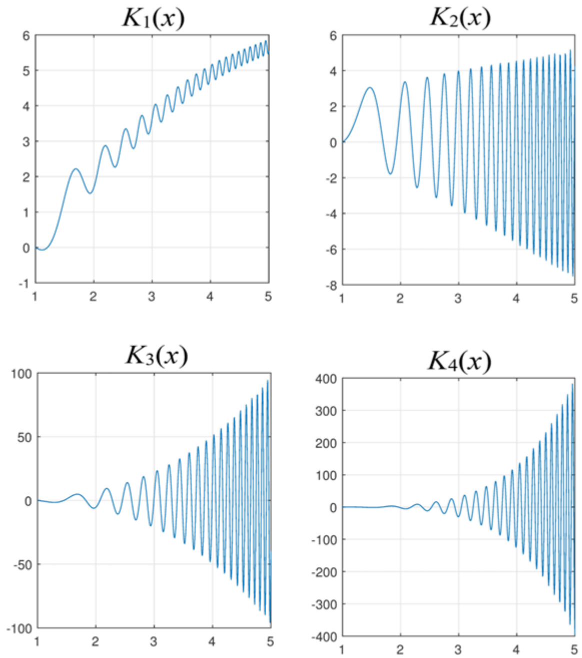

Fig. 2. $K_1(x), K_2(x), K_3(x)$ and $K_4(x)$.

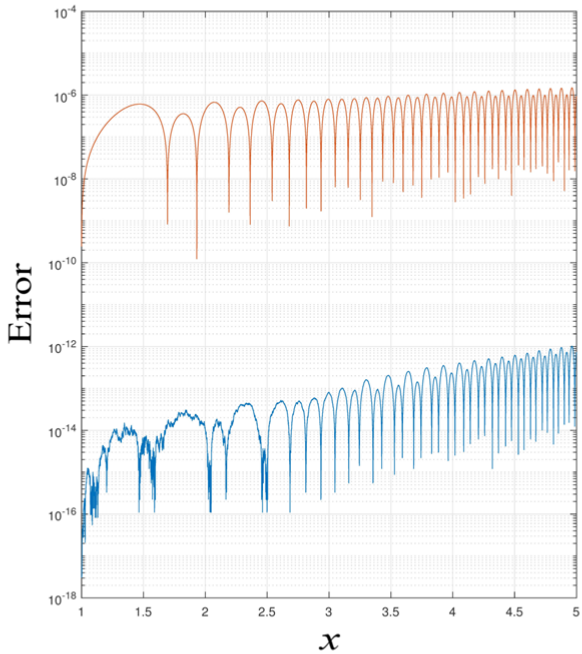

Fig. 3. Error curves for $C(x)$ obtained from both $M_2(x_i, h)$ and $M_4(x_i, h)$, with $h = 6.3 \times 10^{-4}$.

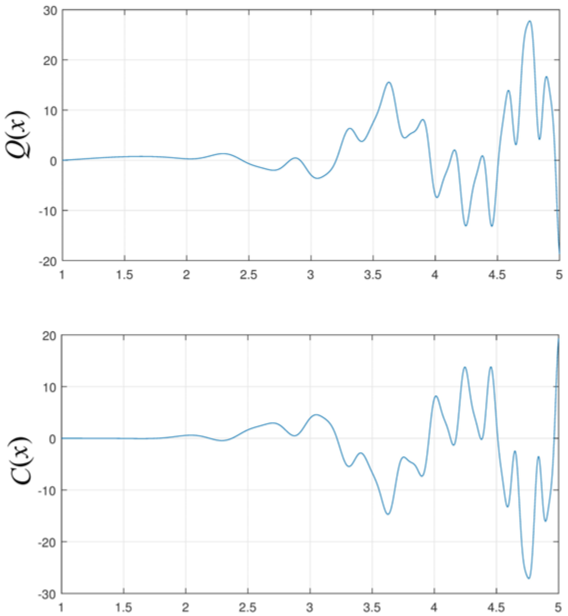

Fig. 4. $Q(x)$ and $C(x)$ for the case of Simpson's Rule.

term $C(x)$ is added to $Q(x)$, the result differs from the true value by $\sim 2 \times 10^{-6}$ (for M_4) and $\sim 5 \times 10^{-9}$ (for M_5) at $x = 5$, and these errors can, of course, be made even smaller by using a smaller stepsize.

Conclusion

We have described a technique by which the task of determining a double integral can be presented as the task of solving a second-order initial value problem. We use Euler's method for this purpose, combined with Richardson extrapolation to yield very accurate results. Indeed, we have achieved accuracy close to machine precision for the numerical example considered here. Moreover, our algorithm lends itself to error control - the stepsize h needed for a desired accuracy can be estimated. Additionally, we have shown how the algorithm can be adapted to yield the error curve for a cubature rule, if such a rule is imposed on the problem.

Future work could address the matter of relative error control, higher-dimensional cubature and the possibility of developing custom Runge-Kutta methods of high order that are able to incorporate the integral term that appears in (3). If the interval of integration is relatively large, it may be wise to divide said interval into subintervals and to apply the algorithm on each subinterval, adding the results in an appropriate manner. Whether or not this is necessary or advantageous needs to investigated. The general case where the limits in the outer integral are arbitrary functions of x should also be considered (we make some comments in this regard in the Appendix).

Lastly, we note that the physical time taken on our platform [27] to compute $C(x)$ – for $\sim$ 6100 nodes – was roughly 30s. To compute the same curve using the symbolic `int(f,a,x)` function in MatLab took as long as an hour. Future research might study this aspect of efficiency in more detail.

Appendix

Richardson extrapolation

We define

$$\begin{aligned} N_0(x,h) \equiv N(x,h) = N\left(x, \frac{h}{2^0}\right) &= M(x) + K_1(x)h + K_2(x)h^2 \\ &\quad + K_3(x)h^3 + K_4(x)h^4 + K_5(x)h^5 + \dots \\ N_1(x,h) \equiv N\left(x, \frac{h}{2}\right) = N\left(x, \frac{h}{2^1}\right) &= M(x) + K_1(x)\frac{h}{2} + K_2(x)\frac{h^2}{4} \\ &\quad + K_3(x)\frac{h^3}{8} + K_4(x)\frac{h^4}{16} + K_5(x)\frac{h^5}{32} + \dots . \end{aligned}$$

and similarly for

$$\begin{aligned} N_2(x,h) &\equiv N\left(x, \frac{h}{2^2}\right) = N\left(x, \frac{h}{4}\right), \\ N_3(x,h) &\equiv N\left(x, \frac{h}{2^3}\right) = N\left(x, \frac{h}{8}\right), \\ N_4(x,h) &\equiv N\left(x, \frac{h}{2^4}\right) = N\left(x, \frac{h}{16}\right), \\ N_5(x,h) &\equiv N\left(x, \frac{h}{2^5}\right) = N\left(x, \frac{h}{32}\right). \end{aligned}$$

For example, to construct a fifth-order method, we seek $\alpha_0, \dots, \alpha_4$ such that

$$M_5(x,h) \equiv \sum_{k=0}^{4} \alpha_k N_k(x,h) = M(x) + O(h^5).$$

It can be shown that we must solve the linear system

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & \frac{1}{2} & \frac{1}{4} & \frac{1}{8} & \frac{1}{16} \\ 1 & \frac{1}{4} & \frac{1}{16} & \frac{1}{64} & \frac{1}{256} \\ 1 & \frac{1}{8} & \frac{1}{64} & \frac{1}{512} & \frac{1}{4096} \\ 1 & \frac{1}{16} & \frac{1}{256} & \frac{1}{4096} & \frac{1}{65536} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which gives

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{315} \\ -\frac{2}{21} \\ \frac{8}{9} \\ -\frac{64}{21} \\ \frac{1024}{315} \end{bmatrix}.$$

In general, we have, for an mth order method

$$M_m(x,h) \equiv \sum_{k=0}^{m-1} \alpha_k N_k(x,h) = M(x) + O(h^m),$$

$$\begin{bmatrix} A_{1,1} & \cdots & \cdots & \cdots & A_{1,m} \\ \vdots & \ddots & & & \vdots \\ \vdots & & A_{i,j} & & \vdots \\ \vdots & & & \ddots & \vdots \\ A_{m,1} & \cdots & \cdots & \cdots & A_{m,m} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \vdots \\ \vdots \\ \vdots \\ \alpha_{m-1} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix} \quad \text{where } A_{i,j} = \left(\frac{1}{2^{i-1}}\right)^{j-1}.$$

Using this, we can find coefficients for 2nd-, 3rd- and 4th-order methods

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ -2 \\ \frac{8}{3} \end{bmatrix}, \quad \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{21} \\ \frac{2}{3} \\ -\frac{8}{3} \\ \frac{64}{21} \end{bmatrix}$$

and a 6th-order method

$$\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{9765} \\ \frac{2}{315} \\ -\frac{8}{63} \\ \frac{64}{63} \\ -\frac{1024}{315} \\ \frac{32768}{9765} \end{bmatrix}.$$

Simpson cubature

A cubature expression, in terms of x, obtained by applying Simpson's Rule to (5), is

$$Q(x) = \frac{(x-1)}{360}\left(18\sin(2) + 18\sin\left(\frac{1}{5}\right) + 72\sin\left(\frac{11}{10}\right)\right)$$
$$+ \frac{(x-1)}{36}\left(x^2 - \frac{x}{5} + 1\right)\begin{pmatrix} \sin(x(x^2+1)) + \sin\left(\frac{x^2}{5}\right) \\ + 4\sin\left(x\left(\frac{x^2}{2} + \frac{x}{10} + \frac{1}{2}\right)\right)\end{pmatrix}$$
$$+ \frac{(x-1)}{36}\left(\left(\frac{x}{2}+\frac{1}{2}\right)^2 - \frac{x}{10} + \frac{9}{10}\right)\begin{pmatrix} 16\sin\left(\left(\frac{x}{2}+\frac{1}{2}\right)\left(\frac{x}{20} + \frac{\left(\frac{x}{2}+\frac{1}{2}\right)^2}{2} + \frac{11}{20}\right)\right) \\ + 4\sin\left(\left(\frac{x}{2}+\frac{1}{2}\right)\left(\frac{x}{10}+\frac{1}{10}\right)\right) \\ + 4\sin\left(\left(\frac{x}{2}+\frac{1}{2}\right)\left(\left(\frac{x}{2}+\frac{1}{2}\right)^2+1\right)\right)\end{pmatrix}.$$

This expression was obtained using symbolic software [27], and includes the effect of the variable limits in the inner integral.

General limits

Consider the more general case

$$C(x) \equiv \int_{a(x)}^{b(x)} \int_{t_0(x)}^{t_1(x)} f(x,t)\,dt\,dx \equiv \int_{a(x)}^{b(x)} I(x)\,dx,$$

where the limits in the outer integral are both functions of x, not necessarily linear, and we have implicitly defined $I(x)$.

We find, dropping the argument of $a(x)$ and $b(x)$ for notational convenience,

$$C'(x) = I\big(b(x)\big)b'(x) - I\big(a(x)\big)a'(x),$$
$$C''(x) = I\big(b(x)\big)b''(x) - I\big(a(x)\big)a''(x) + \frac{dI(b)}{db}b'(x)b'(x) - \frac{dI(a)}{da}a'(x)a'(x). \tag{7}$$

Now,

$$\frac{dI(b)}{db} = \frac{d}{db}\left(\int_{t_0(b)}^{t_1(b)} f(b,t)\,dt\right)$$
$$= f\big(b, t_1(b)\big)\frac{dt_1(b)}{db} - f\big(b, t_0(b)\big)\frac{dt_0(b)}{db} + \int_{t_0(b)}^{t_1(b)} \frac{\partial f(b,t)}{\partial b}\,dt$$

and

$$\frac{dI(a)}{da} = \frac{d}{da}\left(\int_{t_0(a)}^{t_1(a)} f(a,t)\,dt\right)$$
$$= f\big(a, t_1(a)\big)\frac{dt_1(a)}{da} - f\big(a, t_0(a)\big)\frac{dt_0(a)}{da} + \int_{t_0(a)}^{t_1(a)} \frac{\partial f(a,t)}{\partial a}\,dt.$$

There are two approaches we may take to solving (7). The first is the case where a specific value of x is given, say $x = p$. We then simply use p to determine the numerical values for the outer limits, giving

$$C(x) \equiv \int_{a(p)}^{b(p)} \int_{t_0(x)}^{t_1(x)} f(x,t)dtdx.$$

This has the same form as (5), and may be handled in the same way. The second case is when we desire $C(x)$ over an interval, say $[x_0, x_n]$. In this case, we solve the system

$$C' = Z,$$
$$Z' = I(b(x))b''(x) - I(a(x))a''(x) + \frac{dI(b)}{db}b'(x)b'(x) - \frac{dI(a)}{da}a'(x)a'(x)$$

with initial values

$$C(x_0) = \int_{a(x_0)}^{b(x_0)} I(x_0)dx = (b(x_0) - a(x_0))I(x_0),$$
$$Z(x_0) = I(b(x_0))b'(x_0) - I(a(x_0))a'(x_0).$$

We will not investigate this line of research any further here, preferring to consider it in future work.

Nevertheless, for the sake of completeness: in our earlier example we had

$$a(x) = x_0 = \text{const},$$
$$b(x) = x$$

so that

$$a' = a'' = 0,$$
$$b' = 1, \quad b'' = 0,$$
$$t_1(b) = t_1(x), \quad t_0(b) = t_0(x),$$
$$t_1(a) = t_1(x), \quad t_0(a) = t_0(x).$$

Substituting these into (7) gives

$$C'(x) = I(b(x))b'(x) - I(a(x))a'(x) = I(x) = \int_{t_0(x)}^{t_1(x)} f(x,t)dt,$$
$$C''(x) = 0 - 0 + \frac{dI(x)}{dx}(1)(1) - 0$$
$$= f(x, t_1(x))\frac{dt_1(x)}{dx} - f\big(x, t_0(x)\big)\frac{dt_0(x)}{dx} + \int_{t_0(x)}^{t_1(x)} \frac{\partial f(x,t)}{\partial x}dt,$$

as expected.

References

[1] B. Benouahmane, C. Annie, I. Yaman. Near-minimal cubature formulae on the disk. *IMA Journal of Numerical Analysis*, **39**:297–314, 2019. https://doi.org/10.1093/imanum/drx069.

[2] O. Chernukha, Y. Bilushchak, N. Shakhovska, R. Kulh. A numerical method for computing double integrals with variable upper limits. *Mathematics*, **10**(1):108, 2022. https://doi.org/10.3390/math10010108.

[3] R. Cools. Constructing cubature formulae: The science behind the art. *Acta Numerica*, **1**:1–54, 1997. https://doi.org/10.1017/S0962492900002701.

[4] R. Cools. Monomial cubature rules since Stroud: A compilation – Part 2. *Journal of Computational and Applied Mathematics*, **112**:21–27, 1999. https://doi.org/10.1016/s0377-0427(99)00229-0.

[5] R. Cools. An encyclopaedia of cubature formulas. *Journal of Complexity*, **19**:445–453, 2003. https://doi.org/10.1016/S0885-064X(03)00011-6.

[6] R. Cools, P. Rabinowitz. Monomial cubature rules since Stroud: a compilation. *Journal of Computational and Applied Mathematics*, **48**:309–326, 1993. https://doi.org/10.1016/0377-0427(93)90027-9.

[7] F.A. Costabile, L. Guzzardi. On a class of embedded cubature formulae on the simplex. *Avances en Ciencias e Ingenieria*, **6**:A19–A25, 2014.

[8] H. Engels. *Numerical Quadrature and Cubature*. Academic Press, 1980. [ISBN: 012238850X]

[9] A.C.B.B. Filho. A computational algorithm for the numerical integration of a function of one or more variables. *International Journal of Science and Research*, **7**:923–928, 2018. https://doi.org/10.21275/ART2018801.

[10] J. Glaubitz. Stable high-order cubature formulas for experimental data. *Journal of Computational Physics*, **447**:110693, 2021. https://doi.org/10.1016/j.jcp.2021.110693.

[11] J. Glaubitz. Construction and application of provable positive and exact cubature formulas. *IMA Journal of Numerical Analysis*, **43**:1616–1652, 2022. https://doi.org/10.1093/imanum/drac017.

[12] S. Haber. Numerical evaluation of multiple integrals. *SIAM Review*, **12**:481–526, 1970. https://doi.org/doi.org/10.1137/1012102.

[13] P.C. Hammer, A. Wayne Wymore. Numerical evaluation of multiple integrals I. *Mathematical Tables and Other Aids to Computation*, **11**:59–67, 1957. https://doi.org/10.2307/2002154.

[14] S. Hayakawa. Monte carlo cubature construction. *Japan Journal of Industrial and Applied Mathematics*, **38**:561–577, 2021. https://doi.org/10.1007/s13160-020-00451-x.

[15] P. Keast. Cubature formulas for the sphere and the simplex. *Journal of Computational and Applied Mathematics*, **17**:151–172, 1987. https://doi.org/10.1016/0377-0427(87)90044-6.

[16] V. Keshavarzzadeh, R. M. Kirby, A. Narayan. Numerical integration in multiple dimensions with designed quadrature. *SIAM Journal on Scientific Computing*, **40**:A2033–A2061, 2018. https://doi.org/10.1137/17M1137875.

[17] A.R. Krommer, C.W. Ueberhuber. *Computational Integration*. SIAM, 1998. [ISBN: 0898713749]

[18] J. Mahaffy. Numerical integration and differentiation – multiple integrals. 2010. https://jmahaffy.sdsu.edu/courses/s10/math541/lectures/pdf/week09/lecture.pdf

[19] J.S.C. Prentice. Absolute and relative error control in composite interpolatory quadrature: the cirque algorithm. *Journal of Mathematics Research*, **3**:63–72, 2011. https://doi.org/10.5539/jmr.v3n3p63.

[20] J.S.C. Prentice. Relative error control in bivariate interpolatory cubature. *arXiv.org (Cornell University Library)*, 2011. https://doi.org/10.48550/arXiv.1110.2324.

[21] S.L. Sobolev, V. Vaskevich. *The Theory of Cubature Formulas*. Springer, New York, 1997. [ISBN: 9780792346319]

[22] A. Sommariva, M. Vianello. Numerical cubature on scattered data by radial basis functions. *Computing*, **76**:295–310, 2006. https://doi.org/10.1007/s00607-005-0142-2.

[23] A. Sommariva, M. Vianello. Rbf moment computation and meshless cubature on general polygonal regions. *Applied Mathematics and Computation*, **409**:126375, 2021. https://doi.org/10.1016/j.amc.2021.126375.

[24] A.H. Stroud. *Approximate Calculation of Multiple Integrals*. Prentice-Hall, 1971. [ISBN: 9780130438935]

[25] L.N. Trefethen. Cubature, approximation, and isotropy in the hypercube. *SIAM Review*, **59**:469–491, 2017. https://doi.org/10.1137/16M1066312.

[26] C.W. Ueberhuber. *Numerical Computation 2: Methods, Software, and Analysis*. Springer-Verlag, 1997. [ISBN: 9783540620570]

[27] We utilize the computational platform. Intel i7 9700K CPU, Strix Z390-F mainboard, 48GB DDR4 RAM, Windows 10 64-bit, Matlab R2022b.

REZIUMĖ

Dvigubo integralo įvertinimas naudojant Eilerio metodą ir Richardsono ekstrapoliaciją

J.S.C. Prentice

Transformuojame dvigubą integralą į antros eilės pradinės reikšmės uždavinį, kurį sprendžiame naudodami Eilerio metodą ir Richardsono ekstrapoliaciją. Pavyzdžiui, mes norime pasiekti tikslumą, artimą mašininiam tikslumui ($\sim 10^{-13}$). Pastebime, kad algoritmas yra pajėgus nustatyti kubatūrinės formulės klaidos kreivę, ir mes naudojame šią funkciją nustatant Simpsono kubatūrinės formulės klaidos kreivę. Taip pat pateikiame metodo apibendrinimą išorinio integralo netiesinių rėžių atveju.

Raktiniai žodžiai: kubatūra; dvigubas integralas; Euleris; Richardsono ekstrapoliacija; klaida