LMD konf. darbai, 17–23
1997, Vilnius

Zeta-functions of binary Hermitian forms

E. Gaigalas (VU)

This paper generalizes the theorem, proved in [1], for the case of extension $Q(i,\sqrt{d})/Q(\sqrt{d})$, $d \equiv 2 \bmod 8$.

We denote the ring of integers of the field $K = Q(\sqrt{d})$ by A, and the ring of integers of the field $L = Q(i,\sqrt{d})$ by B. For any $\Delta \in A$, $\Delta > 0$, let $H(\Delta)$ be the set of the positive definite binary Hermitian quadratic forms of discriminant Δ:

$$H(\Delta) = \left\{ f(u,v) \,\middle|\, f(u,v) = a|u|^2 + 2\operatorname{Re} bu\bar{v} + d|v|^2,\ u,v \in C \right\}.$$

Here $a, d \in A$, $b \in B$, $ad - |b|^2 = \Delta$.

For any integer ideal $\mathfrak{A}$ of the ring A we define the set

$$R(\mathfrak{A},\Delta) = \left\{ \lambda + \mathfrak{A}B^2 \,\middle|\, \lambda\bar{\lambda} + \Delta \in \mathfrak{A} \right\}$$

and $r(\mathfrak{A},\Delta) = \operatorname{card} R(\mathfrak{A},\Delta)$.

Finally, we define the zeta-function

$$Z(\Delta,s) = \sum_{\mathfrak{A}} \frac{r(\mathfrak{A},\Delta)}{\left(N_{K/Q}\mathfrak{A}\right)^{1+s}},$$

where summation extends over all nonzero integer ideals of A.

This zeta-function is associated with the set of the positive definite binary Hermitian forms $H(\Delta)$. The zeta-function $Z(\Delta,s)$ has the Euler product expansion

$$Z(\Delta,s) = \prod_{\mathfrak{p}} Z_{\mathfrak{p}}\left(\Delta, (N\mathfrak{p})^{-1-s}\right),$$

where

$$Z_{\mathfrak{p}}(\Delta,x) = \sum_{n=0}^{\infty} r\left(\mathfrak{p}^n,\Delta\right) x^n.$$

Here $\mathfrak{p}$ is a nonzero prime ideal of A. We write $\mathfrak{p}^t \| (\Delta)$, if $\mathfrak{p}^t$ is the exact power of $\mathfrak{p}$ dividing ideal (Δ), and denote a prime rational number by p. We need the explicit calculation of the $r\left(\mathfrak{p}^n,\Delta\right)$.

THEOREM. 1. *Let* $\left(\frac{d}{p}\right) = \left(-\frac{1}{p}\right) = 1$. *Then* $pA = \mathfrak{p}_1\mathfrak{p}_2$ *and*

$$r\left(\mathfrak{p}_i^n, \Delta\right) = \begin{cases} p^{n-1}\big((n+1)p - n\big) & \text{for } 0 \leqslant n \leqslant t, \\ p^{n-1}(t+1)(p-1) & \text{for } n \geqslant t+1. \end{cases} \tag{1}$$

2. *Let* $\left(\frac{d}{p}\right) = 1$, $\left(-\frac{1}{p}\right) = -1$. *Then* $pA = \mathfrak{p}_1\mathfrak{p}_2$ *and*

$$r\left(\mathfrak{p}_i^n, \Delta\right) = \begin{cases} p^{n+\frac{1}{2}\left((-1)^n - 1\right)} & \text{for } 0 \leqslant n \leqslant t, \\ \frac{1}{2}\left(1 + (-1)^t\right)p^{n-1}(p+1) & \text{for } n \geqslant t+1. \end{cases} \tag{2}$$

3. *Let* $\left(\frac{d}{p}\right) = -1$. *Then* $pA = (p)$ *and*

$$r\left(p^n, \Delta\right) = \begin{cases} p^{2n-2}\big((n+1)p^2 - n\big) & \text{for } 0 \leqslant n \leqslant t, \\ p^{2n-2}(t+1)(p^2-1) & \text{for } n \geqslant t+1. \end{cases} \tag{3}$$

4. *Let* $p = 2$. *Then* $2A = \left(2, \sqrt{d}\right)^2$ *and*

$$r\left(\left(2, \sqrt{d}\right)^n, \Delta\right) = \begin{cases} 2^n & \text{for } 0 \leqslant n \leqslant t+1, \\ 2^{n-1}\Big(\big(1 + (-1)^t\big)\big(1 + (-1)^{\Delta_2'}\big) & \\ \quad + \big(1 - (-1)^t\big)\big(1 - (-1)^{\Delta_1'}\big)\Big) & \text{for } n \geqslant t+2. \end{cases} \tag{4}$$

Here $\Delta = \Delta_1 + \Delta_2\sqrt{d}$, $\Delta_i = 2^{\frac{t}{2}}\Delta_i'$, *if* t *is even, and* $\Delta_i = 2^{\frac{t+1}{2}}\Delta_i'$, *if* t *is odd.*

5. *Let* $p \mid d$, $p \neq 2$. *Then* $pA = \left(p, \sqrt{d}\right)^2$ *and*

$$r\left(\left(p, \sqrt{d}\right)^n, \Delta\right) = \begin{cases} p^{n+1} + \left(\frac{n}{2} - 1\right)\left(\left(-\frac{1}{p}\right) + 1\right)(p-1)p^n & \text{for } 0 \leqslant n \leqslant t+1, \ n \equiv 0 \bmod 2, \\ p^n\left(p - \left(-\frac{1}{p}\right)\right) & \text{for } 0 \leqslant n \leqslant t+1, \ n \equiv 1 \bmod 2, \\ p^n\left(\left(-\frac{\Delta_1}{p}\right) + 1\right) & \text{for } t = 0, \\ 0 & \text{for } t = 1, \\ p^n\left(p - \left(-\frac{1}{p}\right)\right) & \text{for } n \geqslant t+2. \end{cases} \tag{5}$$

The proofs of formulae (1)–(3) are similar to those in [1].

4. Let's denote

$$Q = \frac{\sqrt{d}}{2} + i\frac{\sqrt{d}}{2}.$$

Then we may write every number $\lambda \in B$ in the form

$$\lambda = x + y\sqrt{d} + Q\big(u + v\sqrt{d}\big) \quad (x, y, u, v \in Z).$$

We have to count the number of (x, y, u, v) such that $\lambda\bar{\lambda} + \Delta \in (2, \sqrt{d})^n$. The numbers x, y, u, v are given mod 2^m if $n = 2m$. If $n = 2m + 1$, the numbers x, u, are given mod 2^{m+1} and y, v – mod 2^m.

Let $n = 2m$ and $0 \leqslant n \leqslant t$. We wish to count the number of solutions of the system of congruences

$$\begin{cases} \left(x + \frac{dv}{2}\right)^2 + \frac{d}{2}(y+u)^2 + \left(\frac{dv}{2}\right)^2 + \frac{d}{2}y^2 \equiv 0 \bmod 2^m, \\ \left(x + \frac{dv}{2} + y + \frac{u}{2}\right)^2 + (d-1)\left(y + \frac{u}{2}\right)^2 \\ \quad + \left(\frac{dv}{2} + \frac{u}{2}\right)^2 + (d-1)\left(\frac{u}{2}\right)^2 \equiv 0 \bmod 2^m. \end{cases} \tag{6}$$

Let's denote $r(2^m) = r((2, \sqrt{d})^n, \Delta)$ and assume, that (x_0, y_0, u_0, v_0) is the solution of the system

$$\begin{cases} \left(x + \frac{dv}{2}\right)^2 + \frac{d}{2}(y+u)^2 + \left(\frac{dv}{2}\right)^2 + \frac{d}{2}y^2 \equiv 0 \bmod 2^{m-1}, \\ \left(x + \frac{dv}{2} + y + \frac{u}{2}\right)^2 + (d-1)\left(y + \frac{u}{2}\right)^2 \\ \quad + \left(\frac{dv}{2} + \frac{u}{2}\right)^2 + (d-1)\left(\frac{u}{2}\right)^2 \equiv 0 \bmod 2^{m-1}. \end{cases} \tag{7}$$

We replace (x, y, u, v) in (6) by $(x_0 + x_1 \cdot 2^{m-1}, y_0 + y_1 \cdot 2^{m-1}, u_0 + u_1 \cdot 2^{m-1}, v_0 + v_1 \cdot 2^{m-1})$. The numbers x_1, y_1, u_1, v_1 are given mod 2. The system (6) becomes

$$x_0 u_1 + x_1 u_0 \equiv -s \bmod 2, \tag{8}$$

where s is found from the equality

$$\left(x_0 + \frac{dv_0}{2} + y_0 + \frac{u_0}{2}\right)^2 + (d-1)\left(y_0 + \frac{u_0}{2}\right)^2 + \left(\frac{dv_0}{2} + \frac{u_0}{2}\right)^2 + (d-1)\left(\frac{u_0}{2}\right)^2 = s \cdot 2^{m-1}.$$

It is easy to check that $x_0 \equiv u_0 \equiv 0 \bmod 2$. Hence $s \equiv 0 \bmod 2$.

The number of solutions of the system (7) with $s \equiv 0 \bmod 2$ is equal to one fourth of the total number of solutions. Hence the recurrence relation $r(2^m) = 4r(2^{m-1})$ is valid. The elementary counting shows that $r(2) = 4$. This gives $r((2, \sqrt{d})^n, \Delta) = 2^n$.

Let $n = 2m$, $n = t + 1$. Then $\Delta_1 = 2^m \Delta_1'$, $\Delta_2 = 2^{m-1}\Delta_2'$ and $\Delta_2' \equiv 1 \bmod 2$. We need to count the number of solutions of the system of congruences

$$\begin{cases} \left(x + \frac{dv}{2}\right)^2 + \frac{d}{2}(y+u)^2 + \left(\frac{dv}{2}\right)^2 + \frac{d}{2}\left(\frac{u}{2}\right)^2 \equiv 0 \bmod 2^m, \\ \left(x + \frac{dv}{2} + y + \frac{u}{2}\right)^2 + (d-1)\left(y + \frac{u}{2}\right)^2 + \left(\frac{dv}{2} + \frac{u}{2}\right)^2 + (d-1)\left(\frac{u}{2}\right)^2 \\ \quad + 2^{m-1}\Delta_2' \equiv 0 \bmod 2^m. \end{cases} \tag{9}$$

We replace x, y, u, v in (9) by $x_0 + x_1 \cdot 2^{m-1}$, $y_0 + y_1 \cdot 2^{m-1}$, $u_0 + u_1 \cdot 2^{m-1}$, $v_0 + v_1 \cdot 2^{m-1}$ $(x_1, y_1, u_1, v_1 \bmod 2)$. This gives us the congruence

$$x_0 u_1 + x_1 u_0 + \Delta_2' \equiv -s \bmod 2, \tag{10}$$

where s is found from the equality

$$\left(x_0 + \frac{dv_0}{2} + y_0 + \frac{u_0}{2}\right)^2 + (d-1)\left(y_0 + \frac{u_0}{2}\right)^2 + \left(\frac{dv_0}{2} + \frac{u_0}{2}\right)^2 + (d-1)\left(\frac{u_0}{2}\right)^2 + 2^{m-2}\Delta_2' = s \cdot 2^{m-1}.$$

As in previous case we have the recurrence relation $r(2^m) = 4r(2^{m-1})$. It is easy to check that $r(2) = 4$. Hence $r((2, \sqrt{d})^n, \Delta) = 2^n$.

Let $n = 2m$, $n \geqslant t + 2$, $t \equiv 1 \bmod 2$. The system

$$\begin{cases} \left(x + \frac{dv}{2}\right)^2 + \frac{d}{2}(y+u)^2 + \left(\frac{dv}{2}\right)^2 + \frac{d}{2}y^2 + 2^{\frac{t+1}{2}}\Delta_1' \equiv 0 \bmod 2^m, \\ \left(x + \frac{dv}{2} + y + \frac{u}{2}\right)^2 + (d-1)\left(y + \frac{u}{2}\right)^2 + \left(\frac{dv}{2} + \frac{u}{2}\right)^2 + (d-1)\left(\frac{u}{2}\right)^2 \\ \quad + 2^{\frac{t+1}{2}}\Delta_1' + 2^{\frac{t-1}{2}}\Delta_2' \equiv 0 \bmod 2^m \end{cases} \tag{11}$$

has no solutions, if $\Delta_1' \equiv 0 \bmod 2$. If $\Delta_1' \equiv 1 \bmod 2$, it is easy to prove the formula $r((2, \sqrt{d})^n, \Delta) = 2^{n+1}$.

Let $n = 2m + 1$, $0 \leqslant n \leqslant t$. We investigate the system

$$\begin{cases} \left(x + \frac{dv}{2}\right)^2 + \frac{d}{2}(y+u)^2 + \left(\frac{dv}{2}\right)^2 + \frac{d}{2}y^2 \equiv 0 \bmod 2^{m+1}, \\ \left(x + \frac{dv}{2} + 2y + u\right)^2 + (d-4)\left(y + \frac{u}{2}\right)^2 + \left(\frac{dv}{2} + u\right)^2 + (d-4)\left(\frac{u}{2}\right)^2 \\ \quad \equiv 0 \bmod 2^{m+1}. \end{cases} \tag{12}$$

It is easy to verify that the recurrence relation $r\left(2^{m+1}\right) = 4r\left(2^m\right)$ is valid. In addition, $r(2) = 4$. Hence $r\left((2, \sqrt{d})^n, \Delta\right) = 2^n$. The proof of the case $n = t + 1$ is similar.

Let $n \geqslant t+2$, $t \equiv 1 \bmod 2$. We investigate the system

$$\begin{cases} \left(x+\frac{dv}{2}\right)^2+\frac{d}{2}(y+u)^2+\left(\frac{dv}{2}\right)^2+\frac{d}{2}y^2+2^{\frac{t+1}{2}}\Delta_1' \equiv 0 \bmod 2^{m+1}, \\ \left(x+\frac{dv}{2}+2y+u\right)^2+(d-4)\left(y+\frac{u}{2}\right)^2+\left(\frac{dv}{2}+u\right)^2+(d-4)\left(\frac{u}{2}\right)^2 \\ \quad +2^{\frac{t+1}{2}}\left(\Delta_1'+\Delta_2'\right) \equiv 0 \bmod 2^{m+1}. \end{cases} \tag{13}$$

In this case $\Delta_2' \equiv 1 \bmod 2$. In a similar way we obtain the formula

$$r\left((2,\sqrt{d})^n,\Delta\right)=\left(1-(-1)^{\Delta_1'}\right)2^n.$$

Let $n \geqslant t+2$, $t \equiv 0 \bmod 2$. Then $\Delta_1' \equiv 1 \bmod 2$ and $r((2,\sqrt{d})^n,\Delta) = (1+(-1)^{\Delta_2'})\,2^n$.

5. Let $p|d$, $p \neq 2$. We'd like to count the number (x, y, u, v) such that

$$\left(x+y\sqrt{d}+Q\left(u+v\sqrt{d}\right)\right)\left(x+y\sqrt{d}+\overline{Q}\left(u+v\sqrt{d}\right)\right)+\Delta \in \left(p,\sqrt{d}\right)^n.$$

The numbers x, y, u, v are given mod p^m if n is even. If n is odd, the numbers x, u are given mod p^{m+1} and y, v – mod p^m.

Let $n = 2m$, $0 \leqslant n \leqslant t$. We need to count the number of solutions of the system of congruences

$$\begin{cases} \left(x+\frac{dv}{2}\right)^2+\frac{d}{2}(y+u)^2+\left(\frac{dv}{2}\right)^2+\frac{d}{2}y^2 \equiv 0 \bmod p^m, \\ \left(x+\frac{dv}{2}+y+\frac{u}{2}\right)^2+(d-1)\left(y+\frac{u}{2}\right)^2 \\ \quad +\left(\frac{dv}{2}+u\right)^2+(d-1)\left(\frac{u}{2}\right)^2 \equiv 0 \bmod p^m. \end{cases} \tag{14}$$

Let (x_0, y_0, u_0, v_0) be a solution of the system mod p^{m-1}. Replacing x, y, u, v in (14) by $x_0+x_1p^{m-1}$, $y_0+y_1p^{m-1}$, $u_0+u_1p^{m-1}$, $v_0+v_1p^{m-1}$ (x_1, y_1, u_1, v_1 mod p), we transform (14) into the system

$$\begin{cases} 2x_0x_1 \equiv -r \bmod p, \\ 2x_0x_1+2x_0v_1+2x_0y_1+x_0u_1+2y_0x_1+u_0x_1+2y_0v_1+2u_0v_1 \\ \quad \equiv -s \bmod p, \end{cases} \tag{15}$$

where r, s are found from the equalities

$$\left(x_0+\frac{dv_0}{2}\right)^2+\frac{d}{2}\left(y_0+u_0\right)^2+\left(\frac{dv_0}{2}\right)^2+\frac{d}{2}y_0^2=r\,p^{m-1},$$

$$\left(x+\frac{dv_0}{2}+y_0+\frac{u_0}{2}\right)^2+(d-1)\left(y_0+\frac{u_0}{2}\right)^2+\left(\frac{dv_0}{2}+\frac{u_0}{2}\right)^2$$
$$+(d-1)\left(\frac{u_0}{2}\right)^2=s\,p^{m-1}.$$

It is easy to check that $x_0 \equiv 0 \bmod p$. Hence (15) transforms into the system

$$\begin{cases} r \equiv 0 \bmod p, \\ x_1(2y_0 + u_0) + 2v_1(y_0 + u_0) \equiv -s \bmod p. \end{cases} \tag{16}$$

Let's denote $r_1(p^m)$ the number of solutions of the system (14) such that $y_0 \equiv u_0 \equiv 0 \bmod p$, and $r_2(p^m)$ – the number of remainder solutions. We wish to count $r_1(p^m)$. We investigate the system $r \equiv s \equiv 0 \bmod p$. By choosing the solutions such that $y_0 \equiv u_0 \equiv 0 \bmod p^2$ it is easy to obtain the recurrence relation $r_1(p^m) = p^2 r_1(p^{m-1}) + p^r r_2(p^{m-1})$. The recurrence relation $r_2(p^m) = p^2 r_2(p^{m-1})$ is valid for remainder solutions. In addition $r(p) = p^3$ and $r_1(p^2) = p^5 + ((-\frac{1}{p}) + 1)(p-1)p^4$. Hence $r((p, \sqrt{d})^n, \Delta) = p^{n+1} + (\frac{n}{2} - 1)((-\frac{1}{p}) + 1)(p-1)p^n$. The case $n = t + 1$ ($t \neq 1$) is similar.

Let $t = 1$, $n \geqslant 1$. Then the system

$$\begin{cases} \left(x + \frac{dv}{2}\right)^2 + \frac{d}{2}(y+u)^2 + \left(\frac{dv}{2}\right)^2 + \frac{d}{2}y^2 + p\Delta_1' \equiv 0 \bmod p^m, \\ \left(x + \frac{dv}{2} + y + \frac{u}{2}\right)^2 + (d-1)\left(y + \frac{u}{2}\right)^2 \\ \quad + \left(\frac{dv}{2} + \frac{u}{2}\right)^2 + (d-1)\left(\frac{u}{2}\right)^2 + p\Delta_1' + \Delta_2 \equiv 0 \bmod p^m \end{cases} \tag{17}$$

has no solutions, because $\Delta_2 \not\equiv 0 \bmod p$.

If $t = 0$, it is easy to obtain the formula $r((p, \sqrt{d})^n, \Delta) = ((\frac{-\Delta_1}{p}) + 1)\, p^n$.

Let $n \geqslant t+2$, $t \neq 0, 1$. The recurrence relation $r(p^m) = p^2 r(p^{m-1})$ is valid in this case. In addition, $r(p) = p^2(p - (-\frac{1}{p}))$. Hence $r((p, \sqrt{d})^n, \Delta) = p^n(p - (-\frac{1}{p}))$.

Let $n = 2m + 1$, $0 \leqslant n \leqslant t$. We investigate the system

$$\begin{cases} \left(x + \frac{dv}{2}\right)^2 + \frac{d}{2}(y+u)^2 + \left(\frac{dv}{2}\right)^2 + \frac{d}{2}y^2 \equiv 0 \bmod p^{m+1}, \\ \left(x + \frac{dv}{2} + py + \frac{pu}{2}\right)^2 + (d-p^2)\left(y + \frac{u}{2}\right)^2 \\ \quad + \left(\frac{dv}{2} + \frac{pu}{2}\right)^2 + (d-p^2)\left(\frac{u}{2}\right)^2 \equiv 0 \bmod p^{m+1}. \end{cases} \tag{18}$$

Replacing x, y, u, v in (18) by $x_0 + x_1 p^m$, $py_0 + y_1 p^m$, $u_0 + u_1 p^m$, $pv_0 + v_1 p^m$ ($x_1, y_1, u_1, v_1 \bmod p$), we transform the system (18) into the system

$$\begin{cases} 2x_0 x_1 \equiv -r \bmod p, \\ 2x_0 x_1 \equiv -s \bmod p, \end{cases} \tag{19}$$

where r, s are found from the equalities

$$\left(x_0 + \frac{dv_0}{2}\right)^2 + \frac{d}{2}(y_0 + u_0)^2 + \left(\frac{dv_0}{2}\right)^2 + \frac{d}{2}y_0^2 = r\, p^m,$$

$$\left(x_0 + \frac{dv_0}{2} + py_0 + \frac{pu_0}{2}\right)^2 + (d-p^2)\left(y_0 + \frac{u_0}{2}\right)^2 + \left(\frac{dv_0}{2} + \frac{pu_0}{2}\right)^2 + (d-p^2)\left(\frac{u_0}{2}\right)^2 = s\, p^m.$$

It is easy to verity that $x_0 \equiv 0 \bmod p$. Hence (19) transforms into $r \equiv s \equiv 0 \bmod p$. This gives the recurrence relation $r(p^m) = p^2 r(p^{m-1})$. In addition, $r(p) = p - (-\frac{1}{p})$. Hence $r((p, \sqrt{d})^n, \Delta) = p^n(p - (-\frac{1}{p}))$. If $n = t+1$, the proof is similar. If $n \geqslant t+2$, $t = 1$, it is easy to verity that $r((p, \sqrt{d})^n, \Delta) = 0$. If $n \geqslant t+2$, $t \neq 1$, we obtain by similar way the formula $r((p, \sqrt{d})^n, \Delta) = p^n(p - (-\frac{1}{p}))$.

COROLLARY. *The zeta-function* $Z(\Delta, s)$ *converges absolutely for* $\operatorname{Re} s > 1$.

REFERENCES

[1] E. Gaigalas, Zeta-functions of binary Hermitian forms, *Liet. Matem. Rink.*, **33** (2) (1993), 182–192.

Apie Hermito kvadratinių formų dzeta-funkcijas

E. Gaigalas

Paskaičiuoti plėtinio $Q(i, \sqrt{d})/Q(\sqrt{d})$, $d \equiv 2 \bmod 8$ Hermito kvadratinių formų dzeta-funkcijos koeficientai.