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Asymptotical expansions in the local limit theorem

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INTRODUCTION

In [7] we demonstrate, that the local distribution law for the values of additive function f: $\mathbf{N} \to \mathbf{Z}$ on set of prime numbers $\mathbf{P}$ induces a local law on the whole semigroup $\mathbf{N}$. Some local limit theorems were obtained, wich generalized analogical results from the very known paper of J. Kubilius [1]. It was observed, that the main term in this more general case has *additional factor*. This phenomenon holds in local limit theorem, in the local limit theorem with asymtotical expansions and in local theorem of large deviations. Further, in paper [10] we proved analogical theorems for multiplicative functions having local law on $\mathbf{P}$.

Since 1990 E.Manstavičius [2], [4] (with K.-H. Indlekofer, R. Warlimont) began to study commutative semigroups $\mathbf{G}$, which are called *arithmetical semigroups* and were introduced by John Knopfmacher in his monograph [3]. Several problems with a title *Open Questions* are possed in this monograph. We select a problem about *local distribution* of values of additive and multiplicative fuctions defined on such semigroups. The papers [5], [8], [9] demonstrated our results of that sort. It was observed that some well known propositions, valid in the semigroup $\mathbf{N}$ get some new specific features. The purpose of this paper is to continue the investigations of this kind by proving the local limit theorem for additive function defined on $\mathbf{G}$ with asymptotical expressions of main term.

SOME RESULTS, DEFINITIONS AND NOTATIONS

Additive arithmetical semigroup is by definition a free commutative semigroup with identity element 1, generated by a countable subset $\mathbf{P}$ of prime elements p and admitting a completely additive integer valued *degree function* δ: $\mathbf{G} \to \mathbf{N} \cup \{0\}$ such that $\delta(p) \geqslant 1$ for each $p \in \mathbf{P}$ and the following axiom holds.

AXIOM. *Constants* $A > 0$, $q > 1$, *and* $0 \leqslant \nu < 1$ *exist, such that*

$$G(n) := \#\{a \in \mathbf{G};\ \delta(a) = n\} = Aq^n + O(q^{\nu n}).$$

As it was proved in paper [2]

$$\pi(k) := \#\{p \in \mathbf{G};\ \delta(p) = k\} = \frac{q^k}{k}(1 - I(G)(-1)^k) + O(q^{c_0 k}),$$

where $\max\{1/2, \nu\} < c_0 < 1$.

Here, $I(G)$ denotes indicator of exceptional zero of the generating series

$$Z(y) := \sum_{n=0}^{\infty} G(n) y^n.$$

We consider a *class* $A(G, r)$ of additive functions $f\colon \mathbf{G} \to \mathbf{Z}$ satisfying conditions

$$\sum_{\substack{p \in P,\ \delta(p)=k \\ f(p)=l}} 1 =: \pi(k)\bigl(\lambda_l + \rho_l(k)\bigr), \qquad l \in Z,\ k \geqslant 1, \tag{1}$$

where $\lambda_k \in [0, 1]$ are constants, $\rho_l(k) := C_l(k) r^{-1}(k)$ are remainder terms with $r(k) \to \infty$, as $k \to \infty$. Besides,

$$\sum_l \bigl|C_l(k)\bigr| < \infty$$

uniformly in $k \geqslant 1$.

Further, we are using traditional notations from papers [5], [7], [9]:

$$\chi := \chi(z) = \sum_l \lambda_l e^{zl}, \quad E = \sum_l l\lambda_l, \qquad \sigma^2 = \sum_l l^2 \lambda_l, \quad \varphi(u) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2},$$

$$\mu_j(z) = (-1)^{j-1} \chi(z) - 1 - zE, \qquad j = 1, 2; \qquad \lambda = \sqrt{\log n}, \qquad y = \frac{m - E\lambda^2}{\lambda}.$$

t_0, τ_0 denote an arbitrary solutions of the equations

$$\sum_l \lambda_l \sin^2(lt/2) = 0, \qquad \sum_l \lambda_l \cos^2(l\tau/2) = 0, \tag{2}$$

belonging to the interval $(-\pi, \pi]$ respectively. $H(f)$ denotes an *additional factor* in the local limit theorem (see [7], [5]).

In papers [5] and [9] two asymptotic formulas for the frequency

$$\nu_n(m) := \frac{1}{Aq^n} \#\bigl\{a \in G;\ \delta(a) = n,\ f(a) = m\bigr\}$$

were proved.

The zone of nontriviality of the first of these formulas is determined by the function $\varphi(u)$ used in the main term. In fact, it agrees with

$$|m - E\lambda^2| \leqslant (1 - h)\lambda\sigma\sqrt{2\log\lambda},$$

where h is any fixed positive number.

The second formula, proved in [9], in probabilistic terminology gives *local theorem of large deviations.* She is valid in the larger region: $m - E\lambda^2 = o(\lambda^2)$.

If function f from the class $A(G, r)$ satisfies a stronger conditions than in [5], then we can enlarge the zone of nontriviality of asymptotic formula for $\nu_n(m)$ by proving *local limit theorem with asymtotical expansions.*

LOCAL THEOREM WITH ASYMPTOTICAL EXPANSIONS

THEOREM. *If an integer fixed number* $s \geqslant 1$ *exists, such that* $f \in A(G, r)$ *with* $r(k) = (\sqrt{\log(k+1)})^{s+4}$, $\lambda_0 < 1$ *and the series*

$$\sum_{l} |l|^{s+2}\lambda_l, \qquad \sum_{p, j \geqslant 2} \left|f(p^j)\right|^s q^{-j\delta(p)}, \qquad \sum_{l} |l|^s \left|C_l(k)\right| \tag{3}$$

converge (the last one uniformly in $k \geqslant 1$*), then, as* $n \to \infty$, $\nu_n(m)$ *equals*

$$\sum_{t_0} e^{-it_0 m} \sum_{j=0}^{s-1} \frac{P_j(-\varphi, t_0)}{(\lambda\sigma)^{j+1}}$$
$$+ (-1)^n I(G) q\big(AZ'(-q^{-1})\big)^{-1} \sum_{\tau_0} e^{-i\tau_0 m} \sum_{j=0}^{s-1} \frac{Q_j(-\varphi, \tau_0)}{(\lambda\sigma)^{j+1}} + O(\lambda^{-s-1})$$

where $P_j(u, t_0)$ *and* $Q_j(\nu, \tau_0)$ *are polynomials of degree* $3j$ *with coefficients depending only on the function* f *and numbers* t_0, τ_0 *respectively.* $P_j(-\varphi, t_0)$ *and* $Q_j(-\varphi, \tau_0)$ *are obtained from* $P_j(u, t_0)$, $Q_j(\nu, \tau_0)$ *by replacing all powers* u^l, ν^l *by* $\varphi^{(l)}(-y/\sigma)$. *Besides, we observe, that*

$$\sum_{t_0} e^{-it_0 m} P_0(-\varphi, t_0) + (-1)^n I(G) q\big(AZ'(-q^{-1})\big)^{-1} \sum_{\tau_0} e^{-i\tau_0 m} Q_0(-\varphi, \tau_0)$$
$$= \varphi\left(\frac{y}{\sigma}\right) H(f).$$

The proof of the theorem is based upon the following lemma.

LEMMA 1. *If* $f \in A(G, r)$, $r(k) = (\sqrt{\log(k+1)})^{s+4}$, *then, as* $n \to \infty$,

$$\frac{1}{Aq^n} \sum_{\delta(a)=n} e^{itf(a)} = \frac{(An)^{\chi-1}}{\Gamma(\chi)} \prod_{p \in P} \left(1 - \frac{1}{\|p\|}\right)^{\chi} \sum_{j=0}^{\infty} \frac{\exp\{itf(p^j)\}}{\|p\|^j}$$
$$+ I(G) \frac{(-1)^n A_1^{\chi} n^{-\chi-1}}{A\Gamma(-\chi)} \prod_{p \in P} \left(1 - \frac{(-1)^{\delta(p)}}{\|p\|}\right)^{\chi}$$
$$\times \sum_{j=0}^{\infty} \frac{(-1)^{j\delta(p)} \exp\{itf(p^j)\}}{\|p\|^j} + O\big(\lambda^{-s-1}\big)$$

uniformly for any $t \in \mathbf{R}$. *Here* $\Gamma(z)$ *denotes the Euler gamma-function,*

$$A_1 := Z'(-q^{-1})q^{-1} \neq 0, \qquad \|a\| := q^{\delta(a)}.$$

Proof see in [6, p. 332, Theorem 1].

Using the result of lemma 1 and following the papers [1], [7] and [5], we obtain, at first,

$$\nu_n(m) = \frac{1}{2\pi A q^n}\int_{-\pi}^{\pi} e^{-itm}\sum_{\delta(a)=n} e^{itf(a)}dt := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-itm}\frac{(An)^{\chi(it)-1}}{\Gamma(\chi(it))}h_1(t)\,dt$$

$$+\frac{I(G)(-1)^n}{2\pi A}\int_{-\pi}^{\pi} e^{-i\tau m}\frac{A_1^{\chi(it)}n^{-\chi(i\tau)-1}}{\Gamma(-\chi(i\tau))}h_2(\tau)\,d\tau + O(\lambda^{-s-1})$$

$$:= J_1 + J_2 + O(\lambda^{-s-1}),$$

where (see [5])

$$J_1 := \sum_{t_0} J_1(t_0), \qquad J_2 := \sum_{\tau_0} J_2(\tau_0).$$

Then, after the substitutions $t \to t + t_0$, $\tau \to \tau + \tau_0$, the path of integration about each of the *saddle points* t_0, τ_0 becomes some neighbourhood of the zero point, say $T(0)$ or $D(0)$ respectively. If there are no solutions for the second equation (2), or if $I(G) = 0$, then $J_2 = 0$.

Now, with the notations

$$L_1(t+t_0) := \frac{A^{\chi(it)-1}}{\Gamma(\chi(it))}h_1(t+t_0), \qquad L_2(\tau+\tau_0) := \frac{A_1^{-\chi(i\tau)}}{\Gamma(\chi(i\tau))}h_2(\tau+\tau_0),$$

we have

$$J_1(t_0) = \frac{e^{-it_0 m}}{2\pi}\int_{T(0)} L_1(t+t_0)\exp\{\mu_1(it)\lambda^2 - ity\lambda\}dt.$$

and

$$J_2(\tau_0) = \frac{I(G)(-1)^n e^{-i\tau_0 m}}{2\pi A}\int_{D(0)} L_2(\tau+\tau_0)\exp\{\mu_1(i\tau)\lambda^2 - i\tau y\lambda\}d\tau.$$

Let's afterwards $\varepsilon > 0$ denote *sufficiently small number*. Using the condition (3), we obtain the estimation

$$\left|\exp\{\mu_1(iu)\lambda^2\}\right| \leqslant \exp\{-c(\varepsilon)\lambda^2\}$$

for $|u| \geqslant \varepsilon$, with some $c(\varepsilon) > 0$. Also, we can represent, for example, integral $J_2(\tau_0)$ in the form

$$J_2(\tau_0) = \frac{I(G)(-1)^n e^{-i\tau_0 m}}{2\pi A}\int_{|\tau|\leqslant\varepsilon} L_2(\tau+\tau_0)\exp\{\mu_1(i\tau)\lambda^2 - i\tau y\lambda\}d\tau + O(\lambda^{-s-1}). \qquad (*)$$

The main point in the proof of theorem is asymptotical expansion of functions $\log L_1(t+t_0)$, $\log L_2(\tau+\tau_0)$ in powers of (it) and $(i\tau)$, respectively.

LEMMA 2. *For any sufficiently small: t from $T(0)$ and τ from $D(0)$, it is possible to represent the functions $L_1(t+t_0)$, $L_2(\tau+\tau_0)$ in the form:*

$$\log L_1(t+t_0)=\sum_{j=0}^{s-1}\beta_j^{(1)}(t_0)(it)^j+O\big(|t|^s\big)$$

and

$$\log L_2(\tau+\tau_0)=\sum_{j=0}^{s-1}\beta_j^{(2)}(\tau_0)(i\tau)^j+O\big(|\tau|^s\big).$$

Proof. From the conditions (1), (3), the well-known properties of the Γ-function and estimations

$$\chi(iu)-1=\sum_{j=0}^{s-1}\chi_j^{(1)}(iu)^j+O\big(|u|^s\big),$$

we deduce, at first, that

$$\log\Gamma\big(\chi(iu)\big)=\sum_{j=1}^{s-1}\gamma_j^{(1)}(iu)^j+O\big(|u|^s\big), \tag{4}$$

with some coefficients, bounded in neighbourhoods $T(0)$ and $D(0)$.

Then, using the formal equality

$$h_j(w):=\prod_p\psi_{pj}(w)=\prod_p\psi_{pj}(w_0)\exp\left\{\sum_p\log\left(1+\left(\frac{\psi_{pj}(w)}{\psi_{pj}(iw_0)}-1\right)\right)\right\},$$

$j=1,2$ (where $w=t$ or $w=\tau$), the conditions of Theorem, and following the papers [5] and [9], we have for each $p\in\mathbf{P}$, with sufficiently large $\|p\|$

$$\log\left(1+\left(\frac{\psi_{pj}(w)}{\psi_{pj}(iw_0)}-1\right)\right)=\sum_{r=1}^{s-1}A_{rp}^{(j)}(w_0)(iw)^r+O\big(|w|^s\big|A_{sp}^{(j)}(w_0,w)\big|\big), \tag{5}$$

where the series

$$\sum_p\big|A_{rp}^{(j)}(w_0)\big|,\qquad\sum_p\big|A_{sp}^{(j)}(w_0,w)\big|,\quad j=1,2;\ r=1,2,\dots,s-1$$

converge (the last one uniformly in $|w|\leqslant\varepsilon$).

Having in mind the well known properties of exponential and logarithmic functions and using expansions (4) and (5) we obtain the assertion of Lemma 2.

By Cauchy's theorem positive constants $\eta_j=\eta_j(s)$, $j=1,2$ exist, such that

$$\big|\beta_r^{(j)}(u)\big|\leqslant c_j\eta_j^{-r},\quad r=1,2,\dots,s-1;\ j=1,2.$$

Further, we have used calculations method and results from papers [1] and [7]. Lets denote

$$\Phi_n^{(j)}(iu) := \lambda^2 \mu_1\left(\frac{iu}{\lambda}\right) + \frac{(u\sigma)^2}{2} + \log L_j\left(\frac{u}{\lambda} + u_0\right).$$

Using results of Lemma 2, we can represent the functions $\exp\{\Phi_n^{(j)}(iu)\}$, $j = 1, 2$ in the form

$$\exp\left\{\Phi_n^{(1)}(it)\right\} = \sum_{j=0}^{s-1} \frac{P_j(it, t_0)}{\lambda^j} + O\left(\frac{|t|^s}{\lambda^s}(1+t^2)^s \exp\left\{\frac{(t\sigma)^2}{4}\right\}\right),$$

and analogously

$$\exp\left\{\Phi_n^{(2)}(i\tau)\right\} = \sum_{j=0}^{s-1} \frac{Q_j(i\tau, \tau_0)}{\lambda^j} + O\left(\frac{|\tau|^s}{\lambda^s}(1+\tau^2)^s \exp\left\{\frac{(\tau\sigma)^2}{4}\right\}\right),$$

where $P_j(u, t_0)$ and $Q_j(v, \tau_0)$ are polynomials of the degree 3j with coefficients depending only on the function f and numbers t_0, τ_0.

From this we have, for example, that

$$L_2\left(\frac{\tau}{\lambda} + \tau_0\right) \exp\left\{\lambda^2 \mu_1\left(\frac{i\tau}{\lambda}\right)\right\} = \exp\left\{-\frac{(\tau\sigma^2)}{2}\right\} \sum_{j=0}^{s-1} \frac{Q_j(i\tau, \tau_0)}{\lambda^j}$$
$$+ O\left(\frac{|\tau|^s}{\lambda^s}(1+\tau^2)^s \exp\left\{-\frac{(\tau\sigma)^2}{4}\right\}\right).$$

Similar formula holds for the function $L_1(\frac{t}{\lambda} + t_0)$.

Putting these expansions into formulas of kind (*) and using the equality

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} (iu)^j \exp\left\{-\frac{1}{2}u^2 - iu\frac{y}{\sigma}\right\} du = \varphi^{(j)}\left(-\frac{y}{\sigma}\right),$$

we end the proof of Theorem.

Also, we enlarged the region of the validity of local theorem to

$$|m - E\lambda^2| \leqslant (1-h)\lambda\sigma\sqrt{2s\log\lambda}.$$

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Lokalios ribinės teoremos asimptotiniai skleidiniai

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Straipsnyje įrodyta lokalioji ribinė teorema adityviosioms sveikareikšmėms funkcijoms apibrėžtoms virš specialaus aritmetinio pusgrupio **G**. Tiriama tokių funkcijų klasė $A(G, r)$, kuriai priklauso funkcijos pusgrupio G pirminių elementų aibėje **P** turinčios lokalųjį skirstinį. Darbe gautas pagrindinio nario asimptotinis skleidinys tuo išplečiant lokaliosios teoremos galiojimo zoną. Straipsnio rezultatai apibendrina gerai žinomą J. Kubiliaus darbą ir yra natūrali autoriaus ankstesnių tyrimų tąsa.