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Quadratic triangular Bézier patches on quadrics

K. Karčiauskas (VU)

1. INTRODUCTION

A problem of testing if a quadratic triangular Bézier patch is on a quadric has been investigated by many authors [1–3, 6–8]. It was considered from the different points of view. And in each case it could be thought as has been solved. But Farin in [4] still declares it as a problem. We present both geometric and analytic conditions, verifying if a given quadratic triangular Bézier patch defines a quadric or not. In order not to complicate a formulation of main theorem some natural assumptions are made. It is not difficult to show that, if it is necessary, after proper subdivision of initial patch all assumptions can be satisfied. A proof of the theorem is based on classical facts from projective geometry of quadrics and well known results about patches on them, discovered by some authors [1, 2, 6, 8]. We also use results of Lu [7] about patches on degenerate quadrics. Our approach leads to simple formulas for testing if a quadratic triangular Bézier patch defines a quadric. This is in contrast to the previous approaches, when testing was described as an algorithm. Also we present simple formulas for construction of all possible types of quadratic triangular patches on quadrics. Some of them look rather strange, but they still represent natural domains on quadrics. Most of the formulas for constructive approach were derived using MAPLE's power of symbolic computation.

The paper is organized as follows. In Section 2 the notations are introduced and the main theorem is formulated. In Section 3 its analytic version is given. Constructive approach is presented in Sections 4–6. A coordinate system is chosen in these sections for every case so that the formulas and computations are most simple. Some of the presented formulas do not possess an expected symmetry. It is caused by an attempt to get the formulas as rational expressions of input data, since we prefer to use exact arithmetic computations.

2. NOTATIONS AND MAIN THEOREM

We denote by:

$\overline{\mathbf{AB}}$ – a line through **A** and **B**;

$\overline{\mathbf{ABC}}$ – a plane, containing noncolinear points **A**, **B**, **C**;

$\overset{\circ}{\triangle}\mathbf{ABC}$ – a set of inner points of a triangle with vertices **A**, **B**, **C**;

cr(**ABCD**) – a crossratio of four collinear points **A**, **B**, **C**, **D**, defined as in [5].

For a quadratic triangular Bézier patch with control points $\mathbf{P}_{ijk}$ and weights w_{ijk}, $i, j, k \geqslant 0$, $i + j + k = 2$ we set: $\mathbf{P}_0 = \mathbf{P}_{200}$, $\mathbf{P}_1 = \mathbf{P}_{020}$, $\mathbf{P}_2 = \mathbf{P}_{002}$, $w_0 = w_{200}$,

$w_1 = w_{020}$, $w_2 = w_{002}$, $\mathbf{P}_{01} = \mathbf{P}_{10} = \mathbf{P}_{110}$, $\mathbf{P}_{12} = \mathbf{P}_{21} = \mathbf{P}_{011}$, $\mathbf{P}_{20} = \mathbf{P}_{02} = \mathbf{P}_{101}$, $w_{01} = w_{10} = w_{110}$, $w_{12} = w_{21} = w_{011}$, $w_{20} = w_{02} = w_{101}$, $\mathbf{Q}_{ij} = (w_i\mathbf{P}_i + w_{ij}\mathbf{P}_{ij})/(w_i + w_{ij})$.

In the last notation of Farin points $\mathbf{Q}_{ij}$ and later on different indexing letters represent different integers from $\{0, 1, 2\}$. We suppose control points are noncomplanar – otherwise there is nothing to investigate. In this and next section we assume additionaly:

(1) all weights w_i, w_{ij} are positive;

(2) the points $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ are noncolinear;

(3) the points $\mathbf{P}_i$, $\mathbf{P}_j$, $\mathbf{P}_{ij}$ are noncolinear and $\mathbf{P}_k \notin \overline{\mathbf{P}_i\mathbf{P}_j\mathbf{P}_{ij}}$;

(4) the points $\mathbf{P}_i$, $\mathbf{P}_{ij}$, $\mathbf{P}_{ik}$, are noncolinear;

(5) the points $\mathbf{P}_{01}$, $\mathbf{P}_{12}$, $\mathbf{P}_{20}$, are noncolinear and the points $\mathbf{P}_i$, $\mathbf{P}_j$, $\mathbf{P}_{ij}$, $\mathbf{P}_{ik}$, $\mathbf{P}_{jk}$ are noncomplanar.

By $\mathbf{P}$ we denote an intersection point of three planes $\overline{\mathbf{P}_i\mathbf{P}_j\mathbf{P}_{ij}}$ (it may be infinite). An intersection point of three planes $\overline{\mathbf{P}_i\mathbf{P}_{ij}\mathbf{P}_{ik}}$ is denoted by $\mathbf{Q}$ (uniqueness of $\mathbf{Q}$ is equivalent to (5)). It also may be infinite. We also set $\mathbf{R}_{ij} = \overline{\mathbf{P}\mathbf{P}_j} \cap \overline{\mathbf{P}_i\mathbf{P}_{ij}}$, $\mathbf{P}'_{ij} = \overline{\mathbf{Q}\mathbf{P}_{ij}} \cap \overline{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2}$, $\mathbf{Q}'_{ij} = \overline{\mathbf{Q}\mathbf{Q}_{ij}} \cap \overline{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2}$, $\mathbf{R}'_{ij} = \overline{\mathbf{P}_i\mathbf{P}'_{ij}} \cap \overline{\mathbf{P}_k\mathbf{P}_j}$, $\mathbf{S}_k = \overline{\mathbf{Q}\mathbf{P}_k} \cap \overline{\mathbf{P}_i\mathbf{P}_j\mathbf{P}_{ij}}$.

THEOREM 1. *The rational quadratic Bézier triangle defines a quadric if and only if three lines* $\overline{\mathbf{P}_k\mathbf{P}'_{ij}}$ *are concurrent and*

$$\text{(a)}\quad \mathrm{cr}(\mathbf{P}_i\mathbf{R}_{ij}\mathbf{Q}_{ij}\mathbf{P}_{ij})\mathrm{cr}(\mathbf{P}_j\mathbf{R}_{ji}\mathbf{Q}_{ji}\mathbf{P}_{ji}) = 4, \quad (i, j) \in \{(0,1), (1,2), (2,0)\},$$

or

$$\text{(a}'\text{)}\quad \begin{aligned} &\mathrm{cr}(\mathbf{P}_i\mathbf{R}'_{ij}\mathbf{Q}'_{ij}\mathbf{P}'_{ij})\mathrm{cr}(\mathbf{P}_j\mathbf{R}'_{ji}\mathbf{Q}'_{ji}\mathbf{P}'_{ji}) = 4, \quad (i, j) \in \{(0,1), (1,2), (2,0)\},\\ &\mathbf{S}_k \in \overset{\circ}{\triangle}\mathbf{P}_i\mathbf{P}_j\mathbf{P}_{ij} \textit{ for all three or exactly one of the points } \mathbf{S}_0, \mathbf{S}_1, \mathbf{S}_2. \end{aligned}$$

If conditions of the theorem are satisfied an equation of a quadric can be written as follows. Let $L_0 = 0$, $L_1 = 0$, $L_{01} = 0$, $L = 0$ are equations of the planes $\overline{\mathbf{P}_0\mathbf{P}_{01}\mathbf{P}_{20}}$, $\overline{\mathbf{P}_1\mathbf{P}_{12}\mathbf{P}_{01}}$, $\overline{\mathbf{P}_0\mathbf{P}_1\mathbf{Q}}$, $\overline{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2}$ respectively. We set $M = L_0L_1L_{01}^2(\mathbf{P}_2) - L_0(\mathbf{P}_2)L_1(\mathbf{P}_2)L_{01}^2$. Then: if (a) is satisfied the equation of a quadric is $ML^2(\mathbf{P}) - M(\mathbf{P})L^2 = 0$; if (a′) is satisfied the equation is $M = 0$. We can get all information about the quadric from its equation.

If (a) is true then depending on the data we get all possible types of quadrics. But if (a′) is true then a quadric is a cone with a vertex $\mathbf{Q}$ (we consider cylinder as a special case of a cone with an infinite vertex).

The next proposition explains when lines $\overline{\mathbf{P}_k\mathbf{P}'_{ij}}$ are concurrent in coordinate free terms. Let $\mathbf{A}_1 = \overline{\mathbf{P}_1\mathbf{P}_{01}} \cap \overline{\mathbf{P}\mathbf{P}_0}$, $\mathbf{A}_2 = \overline{\mathbf{P}_2\mathbf{P}_{20}} \cap \overline{\mathbf{P}\mathbf{P}_0}$, $\mathbf{A}_{12} = \overline{\mathbf{P}_0\mathbf{P}_{01}\mathbf{P}_{20}} \cap \overline{\mathbf{P}_1\mathbf{P}_2}$.

PROPOSITION. *The lines* $\overline{\mathbf{P}_0\mathbf{P}'_{12}}$, $\overline{\mathbf{P}_1\mathbf{P}'_{20}}$, $\overline{\mathbf{P}_2\mathbf{P}'_{01}}$ *are concurrent if and only if* $\mathbf{P}_{12} \in \overline{\mathbf{P}\mathbf{K}}$, $\mathbf{P}_{12} \neq \mathbf{P}$, $\mathbf{P}_{12} \neq \mathbf{K}$, *where* $\mathbf{K} \in \overline{\mathbf{P}_1\mathbf{P}_2}$ *and* $\mathrm{cr}(\mathbf{K}\mathbf{P}_1\mathbf{P}_2\mathbf{A}_{12}) + \mathrm{cr}(\mathbf{P}\mathbf{A}_1\mathbf{A}_2\mathbf{P}_0) = 0$.

Remark 1. There is no doubt an intersection of an expected quadric and $\overline{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2}$ is a conic. So the condition for the lines $\overline{\mathbf{P}_k\mathbf{P}'_{ij}}$ to be concurrent is nothing else but the theorem of Brianchon for conic.

Remark 2. Suppose not all assumptions are satisfied. Then it is not difficult sometimes to check if the patch defines a quadric or not without further investigation. For example:

assumption (2) is not true, but $\mathbf{P}_0$, $\mathbf{P}_1$,$\mathbf{P}_2$ are distinct – "no";
assumption (2) is not true, but $\mathbf{P}_i = \mathbf{P}_j = \mathbf{P}_{ij}$, $\mathbf{P}_k = \mathbf{P}_i$ – "yes";
$\mathbf{P}_0 = \mathbf{P}_1 = \mathbf{P}_2$ – "no".

3. ANALYTIC VERSION OF THE MAIN THEOREM

In this section we assume $\mathbf{P}$ is a finite point. It is easy to check, that common situation can be reduced to this one. Let $\mathbf{P} = (0; 0; 0)$, $\mathbf{P}_0 = (1; 0; 0)$, $\mathbf{P}_1 = (0; 1; 0)$, $\mathbf{P}_2 = (0; 0; 1)$, $\mathbf{P}_{01} = (a_2; b_2; 0)$, $\mathbf{P}_{12} = (0; a_0; b_0)$, $\mathbf{P}_{20} = (b_1; 0; a_1)$. We set $d_i = a_i + b_i - 1$, $0 \leqslant i \leqslant 2$; $h_{ij} = a_i a_j + b_i b_j$, $g_{ij} = (a_k + b_k)(a_j + b_i - 2a_j b_i) - h_{ij}$, $(i, j) \in \{(0, 1), (1, 2), (2, 0)\}$; $f_0 = b_1 b_2 d_0$, $f_1 = a_0 b_2 d_1$, $f_2 = a_0 a_1 d_2$, $f_{01} = b_2 d_0 d_1 h_{01}/g_{01}$, $f_{12} = a_0 d_1 d_2 h_{12}/g_{12}$, $f_{20} = a_0 a_1 d_2 d_0 h_{20}/b_0 g_{20}$.

THEOREM 2. *Rational quadratic Bézier triangle definies a quadric if and only if* $a_0 a_1 a_2 = b_0 b_1 b_2$ *and*

(a) $w_{ij}^2/w_i w_j = 1/4a_k b_k$, $\quad (i, j) \in \{(0, 1), (1, 2), (2, 0)\}$,

or

$$w_{ij}^2/w_i w_j = f_{ij}^2/f_i f_j, \quad (i, j) \in \{(0, 1), (1, 2), (2, 0)\}, \ f_0 f_1 > 0, \ f_0 f_2 > 0,$$

(a′) *all three or exactly one of the expressions* $f_0 f_{01}$, $f_0 f_{12}$, $f_0 f_{20}$ *are positive.*

4. CONSTRUCTIVE APPROACH TO THE MAIN CASE

Here we construct patches under assumptions (2)-(5) of Section 2. The weights are not necessary positive. We do not care much about their positivity in the following sections too. It is not difficult to get from the presented formulas the conditions when all weights are positive. But we do not discuss here those sometimes boring details. And we denote an equation of a quadric by $F = 0$.

Case 4.1: $\mathbf{P}$ *is finite.*
Let $\mathbf{P} = (0; 0; 0)$, $\mathbf{P}_0 = (1; 0; 0)$, $\mathbf{P}_1 = (0; 1; 0)$, $\mathbf{P}_2 = (0; 0; 1)$, $\mathbf{P}_{01} = (a_2; b_2; 0)$, $\mathbf{P}_{20} = (b_1; 0; a_1)$. Then $\mathbf{P}_{12} = (0, a_0, b_0)$, where $a_0 = r b_1 b_2$, $b_0 = r a_1 a_2$, $r \neq 0$, $r \neq 1/(a_1 a_2 + b_1 b_2)$.

4.1.1. The weights are defined by the formulas $w_0 = b_2$, $w_1 = a_2$, $w_2 = a_0 a_2/b_0$, $w_{01} = 1/2$, $w_{12} = a_2/2b_0$, $w_{20} = a_0 a_2/2b_0 b_1$. They are all positive if and only if all a_i, b_i are positive.

$F = a_0a_1a_2x(x-1)+a_2b_0b_1y(y-1)+a_0a_2b_1z(z-1)+b_0b_1xy+a_0a_2xz+a_2b_1yz.$

4.1.2. We set $w_{ij} = f_i$, $w_{ij} = f_{ij}$, where f_i, f_{ij} are defined in Section 3. The quadric is a cone. $F = M$, where M is derived as explained in Section 2. (Only here and in 4.2.2 we are unable to present a compact formula for F).

Remark. Before investigating the positivity of the weights we can change (if necessary) a sign for any two of w_{ij}. The same is true for all cases.

Case 4.2: **P** *is infinite.*

Let $\mathbf{P}_0 = (0; 0; 0)$, $\mathbf{P}_1 = (1; 0; 0)$, $\mathbf{P}_2 = (0; 1; 0)$, $\mathbf{P}_{01} = (1 - a_2; 0; c_2)$, $\mathbf{P}_{20} = (0; a_1; c_1)$. We denote $d_k = (1 - a_i)(1 - a_j) + a_ia_j$. Then $\mathbf{P}_{12} = (a_0; 1 - a_0; c_0)$, where $a_0 = (1 - a_1)(1 - a_2)/d_0$.

4.2.1. The weights are defined by the formulas $w_0 = a_1(1 - a_2)$, $w_1 = a_1a_2$, $w_2 = (1 - a_1)(1 - a_2)$, $w_{01} = a_1/2$, $w_{12} = d_0/2$, $w_{20} = (1 - a_2)/2$. They all are positive if and only if $0 < a_1 < 1$, $0 < a_2 < 1$.

$F = a_1c_2x(x-1) + (1-a_2)c_1y(y-1) + (c_1(1-a_2) + c_2a_1 - c_0d_0)xy + a_1(2a_2 - 1)xz + (a_2 - 1)(2a_1 - 1)yz + a_1(1 - a_2)z.$

4.2.2. The weights are defined by the formulas $w_0 = (1 - a_1)/c_1$, $w_1 = a_0/c_0$, $w_2 = a_0a_1c_2/(1-a_2)c_0c_1$, $w_{01} = d_2/(c_2d_2 - c_1(1-a_0) - c_0d_1)$, $w_{12} = a_0c_2d_0/(c_0(1 - a_2)(c_0d_0 - c_2(1 - a_1) - c_1a_2))$, $w_{20} = a_0a_1c_2d_1/((1 - a_0)(1 - a_2)c_1(c_1d_1 - c_0(1 - a_2) - c_2a_0))$.

The quadric is a cone, $F = M$.

5. CONSTRUCTION OF SPECIAL PATCHES WITH NONDEGENERATE BOUNDARY CONICS

A patch on quadric with nondegenerate boundary conics can not be constructed using formulas from Section 4 if at least two conics touch each other. So we assume here the points $\mathbf{P}_0$, $\mathbf{P}_{01}$, $\mathbf{P}_{20}$ are collinear, $\mathbf{P}_{01} \neq \mathbf{P}_0$, $\mathbf{P}_{20} \neq \mathbf{P}_0$. In the cases 5.1, 5.2, 5.3 the points $\mathbf{P}_{01}$, $\mathbf{P}_{20}$ are fixed without further restrictions on arbitrary line through $\mathbf{P}_0$, which does not lie in $\overline{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2}$. We also assume $w_0 = w_1 = w_2 = 1$ and w_{01}, w_{20} are arbitrary positive values.

Case 5.1: $\overline{\mathbf{P}_1\mathbf{P}_2\mathbf{P}_{12}} = \overline{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2}$

Let $\mathbf{P}_0 = (0; 0; 0)$, $\mathbf{P}_1 = (1; 0; 0)$, $\mathbf{P}_2 = (0; 1; 0)$, $\mathbf{P}_{01} = (0; 0; c_2)$, $\mathbf{P}_{20} = (0; 0; c_1)$ Then $\mathbf{P}_{12} = (a_0; b_0; 0)$, $w_{12} = 1/2rc_1c_2w_{01}w_{20}$, where $a_0 = rc_1^2w_{20}^2$, $b_0 = rc_2^2w_{01}^2$.

$F = b_0x(x-1) + a_0y(y-1) + (a_0 + b_0)z^2/4(c_2^2w_{01}^2 + c_1^2w_{20}^2) + xy + b_0xz/c_2 + a_0yz/c_1.$

Case 5.2: $\overline{\mathbf{P}_1\mathbf{P}_2\mathbf{P}_{12}} \neq \overline{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2}$, **P** *is finite.*

Let $\mathbf{P} = (0; 0; 0)$, $\mathbf{P}_0 = (1; 0; 0)$, $\mathbf{P}_1 = (0; 1; 0)$, $\mathbf{P}_2 = (0; 0; 1)$, $\mathbf{P}_{01} = (a_2; 0; 0)$, $\mathbf{P}_{20} = (a_1; 0; 0)$. We set $d = a_2(1 - a_2)w_{01}^2 + a_1(1 - a_1)w_{20}^2 + w_{01}^2w_{20}^2(a_1 - a_2)^2$,

$b_0 = a_1(1-a_1)w_{20}^2/d$, $c_0 = a_2(1-a_2)w_{01}^2/d$. Then $\mathbf{P}_{12} = (0; b_0, c_0)$, $w_{12} = d/w_{01}w_{20}(a_1+a_2-2a_1a_2)$. The quadric is a cone. If one of a_1, a_2 is zero we get a patch with one boundary conic touching two others.

$F = x^2 + (a_2s_2+1)y^2 + (a_1s_1+1)z^2 + (s_2+2)xy + (s_1+2)xz + (a_2s_2+a_1s_1 - 4w_{01}^2w_{20}^2(a_1-a_2)^2+2)yz - 2x - (a_2s_2+2)y - (a_1s_1+2)z + 1$, where $s_1 = 4w_{20}^2(a_1-1)$, $s_2 = 4w_{01}^2(a_2-1)$.

Case 5.3: $\overline{\mathbf{P}_1\mathbf{P}_2\mathbf{P}_{12}} \neq \overline{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2}$, $\mathbf{P}$ *is infinite.*

Let $\mathbf{P}_0 = (0;0;0)$, $\mathbf{P}_1 = (1;0;0)$, $\mathbf{P}_2 = (0,1,0)$, $\mathbf{P}_{01} = (0;0;c_2)$, $\mathbf{P}_{20} = (0;0;c_1)$. We set $d = c_1w_{20}^2 + c_2w_{01}^2$, $a_0 = c_2w_{01}^2/d$, $c_0 = w_{01}^2w_{20}^2(c_1-c_2)^2/d$. Then $\mathbf{P}_{12} = (1-a_0; a_0; c_0)$, $w_{12} = d/(c_1+c_2)w_{01}w_{20}$. The quadric is a cone.

$F = c_2^2w_{01}^2x(x-1) + c_1^2w_{20}^2y(y-1) + z^2/4 + (c_2^2w_{01}^2 + c_1^2w_{20}^2 - w_{01}^2w_{20}^2(c_1 - c_2)^2)xy + c_2w_{01}^2xz + c_1w_{20}^2yz$.

Case 5.4: $\mathbf{P}_{01}$, $\mathbf{P}_{20} \in \overline{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2}$

$\mathbf{P}_{12}$ is an arbitrary point not lying in $\overline{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2}$. Let $\mathbf{P}_0 = (0;0;0)$, $\mathbf{P}_1 = (1;0;0)$, $\mathbf{P}_2 = (0;1;0)$, $\mathbf{P}_{12} = (1-a_0; a_0; c_0)$. $\mathbf{P}_{01}(a_2; b_2; 0)$ is arbitrary point in $\overline{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2}$, $\mathbf{P}_{01} \notin \overline{\mathbf{P}_0\mathbf{P}_1} \cup \overline{\mathbf{P}_1\mathbf{P}_2} \cup \overline{\mathbf{P}_2\mathbf{P}_0}$, satisfying $a_2 + b_2 \neq 1/2$. w_{12} is any nonzero value. Then $\mathbf{P}_{20} = (a_1; b_1; 0)$, $w_0 = 1 - a_2 - b_2$, $w_1 = a_2$, $w_2 = -b_2$, $w_{01} = 1/2$, $w_{20} = 1/2 - a_2 - b_2$, where $a_1 = a_2/(2a_2+2b_2-1)$, $b_1 = b_2/(2a_2+2b_2-1)$. The quadric is a cone.

$F = b_2c_0w_{12}x(x-1) - a_2c_0w_{12}y(y-1) + (1-a_2-b_2)((w_{12}(c_0-b_2)+a_2b_2)^2 + 4w_{12}^2a_2b_2a_0(1-a_0))z^2/4a_2b_2c_0w_{12} + (1-2a_2)c_0w_{12}xy + (w_{12}(2a_0a_2+2a_0b_2-a_0-c_0) - a_2b_2)xz + (w_{12}(a_2+b_2+a_0-c_0-1) - a_2b_2)yz + (w_{12}(c_0-a_0a_2-a_0b_2)+a_2b_2)z$.

6. CONSTRUCTION OF PATCHES WITH THE DEGENERATE BOUNDARY

In the cases 6.1, 6.2, 6.3 we assume $\mathbf{P}_0$, $\mathbf{P}_1$, $\mathbf{P}_2$ are distinct, in 6.4, 6.5, 6.6 – $\mathbf{P}_1 = \mathbf{P}_2$.

Case 6.1: *the boundary contains two nontangent conics and a line segment.*

Points $\mathbf{P}_{01}$,$\mathbf{P}_{20} \notin \overline{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2}$ are fixed assuming $\mathbf{P}_0$, $\mathbf{P}_{01}$, $\mathbf{P}_{20}$ are noncollinear. Let $\mathbf{P}_0 = (0;0;0)$, $\mathbf{P}_1 = (1;0;0)$, $\mathbf{P}_2 = (0;1;0)$, $\mathbf{P}_{01} = (a_2;0;c_2)$, $\mathbf{P}_{20} = (0;a_1;c_1)$. Then for fixed t we set $w_0 = a_1a_2t$, $w_1 = a_1((1-a_2)t - c_2)$, $w_3 = a_2((1-a_1)t - c_1)$, $w_{01} = a_1t/2$, $w_{20} = a_2t/2$, $w_{12} = (w_1+w_2)/2$,$a_0 = w_1/(w_1+w_2)$, $\mathbf{P}_{12} = (1 - a_0; a_0; 0)$.

$F = a_1c_2x(x-1) + a_2c_1y(y-1) - a_1a_2z^2/t + (a_2c_1 + a_1c_2)xy + a_1(1-2a_2)xz + a_2(1-2a_1)yz + a_1a_2z$.

If $c_2(1-a_1) = c_1(1-a_2)$ (the points $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_{01}$, $\mathbf{P}_{20}$ are complanar) the quadric is a cone. And since $(0;0;t)$ lies on the quadric it is not difficult to understand how changing t we can change a shape of the patch.

Case 6.2: *the boundary contains two tangent conics and a line segment.*

Points $\mathbf{P}_{01}$,$\mathbf{P}_{20} \notin \overline{\mathbf{P}_0\mathbf{P}_1\mathbf{P}_2}$ are fixed assuming $\mathbf{P}_0$, $\mathbf{P}_{01}$, $\mathbf{P}_{20}$ are collinear. Let $\mathbf{P}_0 = (0;0;0)$, $\mathbf{P}_1 = (1;0;0)$, $\mathbf{P}_2 = (0;1;0)$, $\mathbf{P}_{01} = (0;0;c_2)$, $\mathbf{P}_{20} = (0;0;c_1)$. Then $\mathbf{P}_{12} = (1-a_0; a_0; 0) \in \overline{\mathbf{P}_1\mathbf{P}_2}$, $w_0 = w_1 = w_2 = 1$, $w_{12} = d/2c_1c_2w_{01}w_{20}$, where $d = c_2^2w_{01}^2 + c_1^2w_{20}^2$, $a_0 = c_2^2w_{01}^2/d$

$F = c_2^2 w_{01}^2 x(x-1) + c_1^2 w_{20}^2 y(y-1) + z^2/4 + dxy + c_2 w_{01}^2 xz + c_1 w_{20}^2 yz.$
If $c_1 = c_2$ ($\mathbf{P}_{01} = \mathbf{P}_{20}$) the quadric is a cone.

Case 6.3:*the boundary contains two line segments and a conic.*
We assume $\mathbf{P}_{12} \notin \overline{\mathbf{P}_1\mathbf{P}_2}$, w_{12} is arbitrary. Let $\mathbf{P}_0 = (0;0;0)$, $\mathbf{P}_1 = (1;0;0)$, $\mathbf{P}_2 = (0;1;0)$, $\mathbf{P}_{12} = (1-a_0, a_0, c_0)$. Then $\mathbf{P}_{01} = (a_2;0;0) \in \overline{\mathbf{P}_0\mathbf{P}_1}$, $\mathbf{P}_{20} = (0;a_1;0) \in \overline{\mathbf{P}_0\mathbf{P}_2}$, $w_0 = w_1 = w_2 = 1$, $w_{01} = (4t_2^2+1)/4t_2$, $w_{20} = (4t_1^2+1)/4t_1$, where $a_2 = 4t_2^2/(4t_2^2+1)$, $a_1 = 4t_1^2/(4t_1^2+1)$.
$F = (a_0(1-a_0)w_{12}^2 - 1/4)z^2 + c_0^2 w_{12}^2 xy - a_0 c_0 w_{12}^2 xz + c_0(a_0-1)yz + 2c_0 t_1 t_2 w_{12}^2 (x+y-1)z.$

Case 6.4: *the boundary contains two conics.*
In this case a patch always defines quadric. We assume $\mathbf{P}_1 = \mathbf{P}_2 = \mathbf{P}_{12}$, $w_0 = w_1 = w_2 = 1$. Let $\mathbf{P}_0 = (0;0;1)$, $\mathbf{P}_1 = \mathbf{P}_2 = \mathbf{P}_{12} = (0;0;0)$, $\mathbf{P}_{01} = (1;0;0)$, $\mathbf{P}_{20} = (0;1;0)$. Then $F = w_{20}^2 x^2 + w_{01}^2 y^2 + 4w_{20}^2 w_{01}^2 z(x+y+z-1) + 2w_{01}w_{20}w_{12}xy.$

Case 6.5: *the boundary contains two conics and a double line segment.*
Let $\mathbf{P}_0 = (1;0;0)$, $\mathbf{P}_1 = \mathbf{P}_2 = (0;0;0)$, $\mathbf{P}_{01} = (0;1;0)$, $\mathbf{P}_{20} = (0;0;1)$. The weights w_{01}, w_{20} are arbitrary values, $w_0 = w_1 = w_2 = 1$. Then $\mathbf{P}_{12} = (0;b;c)$,where b is arbitrary, and w_{12} are fixed setting
(1) $w_{12} = w_{01}/(w_{01} + b(w_{20} - w_{01}))$, $c = -bw_{20}/w_{01}$
or
(2) $w_{12} = w_{01}/(-w_{01} + b(w_{20} + w_{01}))$, $c = bw_{20}/w_{01}$.
The quadric is a cone. $F = 4w_{01}^2 w_{20}^2 x(x-1) + w_{20}^2 y^2 + 4w_{01}^2 w_{20}^2 y(x+z) + dz$, where $d = 2w_{01}w_{20}$ for case (1) and $d = -2w_{01}w_{20}$ for case (2).

Case 6.6: $\mathbf{P}_{01} = \mathbf{P}_{20}$
Let $\mathbf{P}_0 = (1;0;0)$, $\mathbf{P}_1 = \mathbf{P}_2 = (0;0;0)$, $\mathbf{P}_{01} = \mathbf{P}_{20} = (0;1;0)$, $\mathbf{P}_{12} = (0;0;1)$. The weights w_{01}, w_{12} are arbitrary values and $w_0 = w_1 = w_2 = 1$. We set
(1) $w_{20} = w_{01}$
or
(2) $w_{20} = -w_{01}$.
The quadric is a cone. $F = 4w_{01}^2 w_{12}^2 x(x+y-1) - w_{12}^2 y^2 + dxz$, where $d = 4w_{01}^2(w_{12}-1)$ for case (1) and $d = 4w_{01}^2(w_{12}+1)$ for case (2).

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Kvadrikių trikampės kvadratinės Bezje skiautės

K. Karčiauskas

Straipsnyje formuojamos būtinos ir pakankamos sąlygos, nusakančios, kada trikampė kvadratinė racionali Bezje skiautė priklauso kvadrikei. Be to pateikiamos formulės, įgalinančios efektyviai konstruoti šias skiautes. Konstruktyvios formulės gautos pasinaudojus MAPLE paketo simbolinio skaičiavimo galimybėmis.