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Indexed multi-succedent calculus with invertible rules for the constructive logic

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1. INTRODUCTION

Common sequent calculus of the constructive predicate logic is one-succedent, i.e., it deals with the following objects: $A_1, \ldots, A_n \to \Theta$, here $\Theta \in \{\varnothing, A\}$. Besides one-succedent calculus, a multi-succedent calculus of the constructive predicate logic is investigated, see [1], [2]. Let us denote it by LJ^*. It is quite natural that the law of the exluded middle $A \vee \neg A$ and the double nagation law are not logically valid in LJ^*. One-succedent sequent calculus has non-invertible rules and LJ^* has ones as well. For instance, the left premiss of the LJ^* logical rule

$$\frac{A \supset B, \Gamma \to A;\ B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}(\supset\to)$$

is not derivable when $\Gamma = \varnothing$, $\Delta = (A \supset B)$, though it is obvious that the conclusion of the same rule under such conditions is an axiom. Or, let us say, the premiss of the rule

$$\frac{\neg A, \Gamma \to A}{\neg A, \Gamma \to \Delta}(\neg\to)$$

is not derivable when $\Gamma = B$, $\Delta = \neg A$, but the conlusion of the rule with $\Gamma = B$, $\Delta = \neg A$ is derivable.

One can give similar examples for the following LJ^* rules: $(\to\supset)$, $(\to\neg)$, $(\to\forall)$. Thus, the derivability of conclusions of the rules $(\supset\to)$, $(\to\supset)$, $(\neg\to)$, $(\to\neg)$, $(\to\forall)$ does not imply the derivability of the premisses. The fact is very inconvenient for automatic proof search.

A calculus O with all invertible rules is constructed in [4]. The calculus O is equivalent to $G3$ (see [3]) with respect to one-succedent sequents.

The purpose of the paper is to construct a multi-succedent calculus O^* with all invertible rules for the constructive predicate logic.

2. THE CALCULUS O^*

Now let us go to the construction of O^*. The objects of investigations here are multi-set generalized sequents:

$$[I_1|F_1], \ldots, [I_m|F_m] \to [I_{m+1}|F_{m+1}], \ldots, [I_n|F_n] \quad (0 \leqslant m \leqslant n),$$

I_i $(1 \leqslant i \leqslant n)$ are used here for finite sets of natural numbers and F_i are used for predicate formulae.

Axiom scheme is

$$\Gamma, [I_1 \mid A] \to [I_2 \mid A], \Delta \quad (I \cap I_2 \neq \varnothing).$$

The concept of sequent derivation is common.

Logical rules are divided into three groups.

2.1. *The first group of O^* rules*

The first group consists of the rules which correspond to one-premiss invertible rules of LJ^*:

$$\frac{\|\Gamma, [I \mid A], [I \mid B] \to \Delta\|}{\Gamma, [I \mid A \wedge B] \to \Delta}(\wedge \to),$$

$$\frac{\|\Gamma \to [I \mid A], [I \mid B], \Delta\|}{\Gamma \to [I \mid A \vee B], \Delta}(\to \vee),$$

$$\frac{\|\Gamma, [I \mid [A]^x_t], [I \mid \forall x A] \to \Delta\|}{\Gamma, [I \mid \forall x A] \to \Delta}(\forall \to),$$

$$\frac{\|\Gamma \to [I \mid [A]^x_t], [I \mid \exists x A], \Delta\|}{\Gamma \to [I \mid \exists x A], \Delta}(\to \exists),$$

$$\frac{\|\Gamma, [I \mid [A]^x_b] \to \Delta\|}{\Gamma, [I \mid \exists x A] \to \Delta}(\exists \to).$$

Let S be a sequent, then $\|S\|$ means simplifications of S of the following two kinds: 1) we remove every formula with an empty set of indices, i.e., every $[|A]$ formula; 2) we contract all indexed formulae with the same predicate formula into one indexed formula, uniting index sets of the formulae, e.g., instead of $[I_1 \mid A], [I_2 \mid A]$ we get $[I \mid A]$, here $I = I_1 \cup I_2$ (of course, it is applied to antecedent and succedent separately).

2.2. *The second group of O^* rules*

The second group consists of the rules which correspond to two-premiss invertible rules of LJ^*:

$$\frac{\|\Gamma \to [n \mid A], [I_1, I_2 \mid A \wedge B], \Delta\|; \|\Gamma \to [n \mid B], [I_1, I_2 \mid A \wedge B], \Delta\|}{\Gamma \to [I_1, n, I_2 \mid A \wedge B], \Delta}(\to \wedge).$$

$$\frac{\|\Gamma, [n \mid A], [I_1, I_2 \mid A \vee B] \to \Delta\|; \|\Gamma, [n \mid B], [I_1, I_2 \mid A \vee B] \to \Delta\|}{\Gamma, [I_1, n, I_2 \mid A \vee B] \to \Delta}(\vee \to).$$

2.3. The third group of O^* rules

The third group consists of the rules which correspond to non-invertible rules of LJ^*:

$$\frac{\|[\Gamma, [I_1, n, I_2 \mid \neg A]]^{n/m} \to [m|A], \Delta\|}{\Gamma, [I_1, n, I_2 \mid \neg A] \to \Delta} (\neg \to),$$

$$\frac{\|[m|A], [\Gamma]^{n/m} \to [I_1, n, I_2 \mid \neg A], \Delta\|}{\Gamma \to [I_1, n, I_2 \mid \neg A], \Delta} (\to \neg),$$

$$\frac{\|[m|A], [\Gamma]^{n/m} \to [m|B], [I_1, n, I_2 \mid A \supset B], \Delta\|}{\Gamma \to [I_1, n, I_2 \mid A \supset B], \Delta} (\to\supset),$$

$$\frac{\|[\Gamma, [I_1, n, I_2|A \supset B]]^{n/m} \to [m|A], \Delta\|; \|\Gamma, [n|B], [I_1, n, I_2|A \supset B] \to \Delta\|}{\Gamma, [I_1, n, I_2 \mid A \supset B] \to \Delta} (\supset\to),$$

$$\frac{\|[\Gamma]^{n/m} \to [m|[A]^x_b], [I_1, n, I_2 \mid \forall x A], \Delta\|}{\Gamma \to [I_1, n, I_2 \mid \forall x A], \Delta} (\to \forall).$$

Letters n, m are used for natural numbers. $[\Gamma]^{n/m}$ stands for the result of adding m to every formulae $[I, n, I'|F]$ in Γ, i.e. from $[I, n, I'|F]$ we get $[I, n, I', m|F]$. n is called a fixed number and m is called the main number. The main number is supposed to be the least natural number which does not occur in the conclusion of the corresponding rule.

3. INVERTIBILITY OF O^* RULES

Let us call a generalized sequent S_1 narrowing of a generalized sequent S_2, if the equality $S_1 = \|S_2'\|$ holds. Here S_2' stands for the result of removing some natural numbers $n_1, \ldots, n_f$ from all index sets of formulae of S_2.

LEMMA 1. *If narrowing of any generalized sequent is derivable, then the original sequent is derivable as well.*

Proof. It is an obvious fact.

The corollary of the lemma is as follows: all rules of the third group are invertible.

LEMMA 2. *All the O^* rules are invertible.*

Proof. From the corollary of lemma 1 we have that the rules $(\neg \to)$, $(\to \neg)$, $(\to\supset)$, $(\supset\to)$, $(\to \forall)$ are invertible. Since the proof of invertibility of the rest O^* rules is very similar to the proof of the corresponding LJ^* rules, we give only a fragment of the proof. We use induction on the height h of derivation of the conclusion. Let us prove, for instance, invertibility of the rule:

$$\frac{\|\Gamma \to [I|A], [I|B], \Delta\|}{\Gamma \to [I|A \vee B], \Delta} (\to \vee).$$

1. Base case: $h = 0$; then $\Gamma = \Gamma_1, [I_1|A \vee B]$, $I \cap I_1 \neq \varnothing$. The derivation of the premiss of the rule is as follows:

$$\frac{\|\Gamma_1, [n|A], [I_1'|A \vee B] \to [I|A], [I|B], \Delta\|; \quad \|\Gamma_1, [n|B], [I_1'|A \vee B] \to [I|A], [I|B], \Delta\|}{\Gamma_1, [I_1|A \vee B] \to [I|A], [I|B], \Delta} (\vee \to).$$

Here $n \in I \cap I_1$ (since $I \cap I_1 \neq \varnothing$, we can choose such n); $I_1' = I_1 \backslash \{n\}$.

2. Let $h > 0$. Then we have:

$$\frac{\dots}{\Gamma \to [I|A \vee B], \Delta} (i),$$

here i is one of the O^* rules. Let i be $(\to \exists)$. Then we have

$$\frac{\|\Gamma \to [I|A \vee B], [I_1|[A]_t^x], [I_1|\exists x C], \Delta\|}{\Gamma \to [I|A \vee B], [I_1|\exists x C], \Delta} (\to \exists).$$

Since $\|\Gamma \to [I|A], [I|B], [I_1|[A]_t^x], [I_1|\exists x C], \Delta\|$ is derivable in O^* by the hypothesis of induction, we can get the derivation of the premiss of the rule $(\to \vee)$ in O^*, i.e.:

$$\frac{\|\Gamma \to [I|A], [I|B], [I_1|[A]_t^x], [I_1|\exists x C], \Delta\|}{\Gamma \to [I|A], [I|B], [I_1|\exists x C], \Delta} (\to \exists).$$

Now let us take $i = (\neg \to)$:

$$\frac{\|[\Gamma, [I_1, n, I_2|\neg D]]^{n/m} \to [m|D], [I|A \vee B], \Delta\|}{\Gamma, [I_1, n, I_2|\neg D] \to [I|A \vee B], \Delta} (\neg \to).$$

$\|[\Gamma]^{n/m}, [I_1, n, I_2, m|\neg D] \to [m|D][I|A], [I|B], \Delta\|$ is derivable by the hypothesis of the induction, therefore:

$$\frac{\|[\Gamma]^{n/m}, [I_1, n, I_2, m|\neg D] \to [m|D][I|A], [I|B], \Delta\|}{\Gamma, [I_1, n, I_2, |\neg D] \to [I|A], [I|B], \Delta} (\neg \to).$$

So we have invertibility of $(\to \vee)$. Other cases, e.g., $i = (\to \supset)$, $i = (\exists \to)$ and so on, are considered similarly. In the same way we get invertibility of all rules of the first and second groups of O^* rules and the right premiss of $(\supset \to)$.

Thus, all the O^* rules are invertible.

4. EQUIVALENCE OF O^* AND LJ^*

We say that a sequent

$$F_1, \dots, F_m \to F_{m+1}, \dots, F_n \quad (0 \leqslant m \leqslant n)$$

is derivable in O^* iff the generalized sequent

$$[1|F_1], \dots, [1|F_m] \to [1|F_{m+1}], \dots, [1|F_n] \quad (0 \leqslant m \leqslant n)$$

is derivable in O^*.

Now we prove the main fact of the paper:

THEOREM. *Any sequent*

$$F_1, \ldots, F_m \to F_{m+1}, \ldots, F_n \quad (0 \leqslant m \leqslant n)$$

is derivable in O^* *iff it is derivable in* LJ^*.

Proof. It is obvious that if a sequent is derivable in LJ^*, it is derivable in O^* as well. It is simple to reconstruct any derivation from LJ^* to O^*. We must only add indices and change LJ^* rules into the corresponding O^* rules. Let us take an example. First take a derivation of the sequent $\to \neg\neg(A \vee \neg A)$ in LJ^*:

$$\begin{array}{cl}
\neg(A \vee \neg A), A \to A, \neg A & \\
\hline
\neg(A \vee \neg A), A \to (A \vee \neg A) & (\to \vee) \\
\hline
\neg(A \vee \neg A), A \to & (\neg \to) \\
\hline
\neg(A \vee \neg A) \to A, \neg A & (\to \neg) \\
\hline
\neg(A \vee \neg A) \to (A \vee \neg A) & (\to \vee) \\
\hline
\neg(A \vee \neg A) \to & (\neg \to) \\
\hline
\to \neg\neg(A \vee \neg A) & (\to \neg).
\end{array}$$

The derivation of the same sequent in O^* looks like that:

$$\begin{array}{cl}
[2,3,4,5|\neg(A \vee \neg A)], [4,5|A] \to [3,5|A], [3,5|\neg A], [1|\neg\neg(A \vee \neg A)] & \\
\hline
[2,3,4,5|\neg(A \vee \neg A)][4,5|A] \to [5|A \vee \neg A], [3|A], [3|\neg A], [1|\neg\neg(A \vee \neg A) & (\to \vee) \\
\hline
[2,3,4|\neg(A \vee \neg A)], [4|A] \to [3|A], [3|\neg A], [1|\neg\neg(A \vee \neg A) & (\neg \to) \\
\hline
[2,3|\neg(A \vee \neg A)] \to [3|A], [3|\neg A], [1|\neg\neg(A \vee \neg A)] & (\to \neg) \\
\hline
[2,3|\neg(A \vee \neg A)] \to [3|A \vee \neg A], [1|\neg\neg(A \vee \neg A)] & (\to \vee) \\
\hline
[2|\neg(A \vee \neg A)] \to [1|\neg\neg(A \vee \neg A)] & (\neg \to) \\
\hline
\to [1|\neg\neg(A \vee \neg A)] & (\to \neg).
\end{array}$$

The proof from O^* to LJ^* is more complex. First we notice that O^* rules of the first and second groups are very similar to the respective LJ^* rules. If we took only those rules, the transformation of derivation from O^* to LJ^* would be very simple. The only thing we should have to do is removing all indices and, in the case of the rules $(\to \wedge)$, $(\vee \to)$, we should have to remove formulae $[I_1, I_2|A \wedge B]$ or $[I_1, I_2|A \vee B]$, respectively, from premisses of the rules.

Now, say we have the rules from group 3, say

$$\frac{\|[m|A], [\Gamma]^{n/m} \to [m|B], [I_1, n, I_2|A \supset B], \Delta\|}{\Gamma \to [I_1, n, I_2|A \supset B], \Delta}\ (\to \supset)$$

in a derivation.

It can be observed that $[m|B]$, $[m|A]$ and $\Delta' = ([I_1, n, I_2|A \supset B], \Delta)$ have index sets whose intersection is an empty set, and even more, the intersection will always be an empty set. It implies that $[m|A], [m|B]$ and Δ' will never be able to give axioms between them. Thus, if

$$\|[m|A], [\Gamma]^{n/m} \to [m|B], [I_1, n, I_2|A \supset B], \Delta\|$$

is derivable, then axioms can be obtained between $[\Gamma]^{n/m}$ and Δ' or between $[\Gamma]^{n/m}$ and $[m|A], [m|B]$. If the first is the case, then the rule $(\to\supset)$ was used in vain, and we can omit it. In the second case, it is possible to remove Δ' from the premiss of $(\to\supset)$, since axioms are obtained between $[\Gamma]^{n/m}$ and $[m|A], [m|B]$, but then the O^* rule $(\to\supset)$ is very similar to the LJ^* rule $(\to\supset)$. We can get the LJ^* rule $(\to\supset)$ by removing indices from the O^* rule $(\to\supset)$. So there is only one possibility from two ones: whether inclusion of $(\to\supset)$ into the derivation was useless or $(\to\supset)$ can easily be transformed into the LJ^* rule $(\to\supset)$. The same holds for the rest third group rules of O^*.

Thus, derivability of a sequent in O^* implies derivability of the same sequent in LJ^* and v.v., and therefore O^* and LJ^* are equivalent.

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Indeksinis daugiasukcedentinis skaičiavimas su apverčiamomis taisyklėmis konstruktyvinei logikai

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Paprastai konstruktyvinėje predikatų logikoje taikomas viensukcedentinis šios logikos skaičiavimas. Jame nagrinėjami objektai yra tokios sekvencijos: $A_1, \ldots, A_n \to \Theta$ $(\Theta \in \varnothing, A)$. Be viensukcedentinio skaičiavimo yra nagrinėjamas daugiasukcedentinis, turintis kai kuriuos privalumus pirmojo atvilgiu. Daugiasukcedentinis (kaip ir veinsukcedentinis) skaičiavimas turi neapverčiamų taisyklių, kurios apsunkina automatinę įrodymo paiešką. Šiame darbe yra sukonstruotas indeksinis predikatų logikos skaičiavimas, kurio visos taisyklės yra apverčiamos ir įrodytas jo ekvivalentiškumas daugiasukcedentiniam konstruktyvinės logikos skaičiavimui.