*LMD konf. darbai,* 268–273
1997, Vilnius

# Synchronizing influence of identical noise in chaotic systems

**F. Ivanauskas, T. Meškauskas (VU), B. Kaulakys (ITPA)**

## INTRODUCTION

Often solutions of the nonlinear dynamical systems are very sensitive to the initial conditions and unpredictable, i.e. the systems are chaotic. It is said that dynamical model is well-posed if there exists a unique solution uniformly depending on initial conditions. Unfortunately in today's nonlinear science applications we observe a lot of phenomena exhibiting an apparent random behavior or *deterministic chaos*. As a simple example one can recall a well-known probabilistic model of a coin, initially placed on its rim. A slight touch is sufficient to determine the falling side of a coin, i.e. we deal with evolution, extremely sensitive to initial state. The recent studies [1, 2] have shown that deterministic chaos is by no means exceptional but a typical property of many nonlinear systems in physics, meteorology, economics and other fields of science.

To illustrate possible properties of such systems let us consider a simple discrete time problem – the so-called logistic map

$$x_{n+1} = r x_n (1 - x_n), \quad n = 1, 2, \dots, \quad x_0 \in [0, 1],$$

where $r$ is a parameter. Logistic map comes from continuous time logistic differential equation arising in some models of mathematical biology. In the case $r = 4$ the solution of above recurrence relations can be written explicitly:

$$x_n = \sin^2 \left(2^n \arcsin \sqrt{x_0}\right).$$

From this expression one can easily conclude unstability to initial value $x_0$.

It becomes practically impossible to predict the long-time behavior of chaotic systems, because in practice one can only fix their initial conditions with finite accuracy, and errors increase exponentially fast. Trying to solve such a system on a computer, the results depends for longer and longer times on more and more digits in the irrational numbers which represents the initial conditions. Since the digits in irrational numbers are irregularly distributed, the evolution becomes chaotic.

It might be expected that when turning on additional random perturbations in such models their behavior becomes even "more chaotic". However, a transition from chaotic to nonchaotic behavior in an ensemble (set) of particles (solutions) with different initial conditions bounded in a fixed external potential and driven by an identical sequence of random forces (perturbations) was observed by Fahy and

Hamann [3]. It has been shown that the ensemble of trajectories in such a case may become identical when $t \to \infty$. The system becomes not chaotic: the trajectories are independent on the initial conditions. The similar effects have been observed in the different systems as well [2, 3] and have resulted to the discussion concerning the origin and causality of such nonchaotic behavior [2, 4].

Moreover, the observed effect resembles a phase transition but does not depend crucially on the dimension of the space in which the particles move. This phenomenon has some importance for Monte Carlo simulations and can influence on the clustering of particles process.

Here we consider nonlinear second order differential equation which, for some parameters values, is unstable according to initial conditions. Following the paper by Fahy and Hamann [3] we introduce some additional stochastics to govern deterministic chaos arising from our differential model. Physically such perturbations correspond to periodical shaking (identical noise) of an ensemble of trajectories moving according to the nonlinear differential rule. We generalize [4] investigating more complex model, which takes into the account the influence of friction and external force, described by the terms $\gamma x'$ and $a \sin t$ in our equation.

Our theoretical analysis is based on the mapping form of equations of motion for the distance between the particles and the difference of the velocity of the particles while numerical calculations are performed according to the derived mapping equations as well as directly calculating the system's trajectories and the Lyapunov exponents (LE). The averaged LE (or Kolmogorov–Sinai entropy, since we deal with one-dimensional map) is the most important measure by which the "degree of chaos" in the system can be evaluated. From the analysis of LE as well as from direct calculations in the phase-space $(x, x')$ we conclude possibility of synchronization in the initially chaotic system. Sufficiently frequent perturbation makes the system not chaotic: the different trajectories become identical when $t \to \infty$, i.e. independent on the initial conditions.

## MATHEMATICAL MODEL OF DYNAMICAL CHAOS AFFECTED BY NOISE

We consider a Cauchy problem for the second order ordinary differential equation:

$$\frac{d^2x}{dt^2} = -\frac{1}{m}\frac{dV(x)}{dx} - \gamma\frac{dx}{dt} + a\sin t, \quad t > 0, \tag{1}$$

$$x(0) = x_0, \tag{2}$$

$$\left.\frac{dx}{dt}\right|_{t=0} = v_0, \tag{3}$$

where $m, a$ and $\gamma \geqslant 0$ are real coefficients. Problem (1)–(3) describes a particle of mass $m$ moving with friction coefficient $\gamma$ in the potential $V(x)$ and affected by periodical external force $a \sin t$. Initial values $x_0$ and $v_0$ represent starting coordinate and velocity, respectively. Model (1)–(3) comes from the well-known 2-nd Newton law.

For simplicity let $m = 1$. We consider the case of Duffing potential

$$V(x) = x^4 - x^2.$$

Then Eq. (1) appears in the following form:

$$\frac{d^2x}{dt^2} = 2x - 4x^3 - \gamma\frac{dx}{dt} + a\sin t, \quad t > 0, \tag{4}$$

Note that already when $\gamma = a = 0$ initial values $x_0 = v_0 = 0$ are unstable. In this case (4), (2), (3) satisfies a class of functions $x(t) = \operatorname{sech}(\sqrt{2}t + \operatorname{arsech} x_0)$ obviously not converging (uniformly) to the trivial solution $x(t) \equiv 0$, when $x_0 \to 0$.

From general theory of differential equations follows, that once initial conditions (2), (3) are defined, there exists a unique solution of Eq. (4) (Cauchy-Kovalevskaya theorem). We choose initial values $x_0$ and $v_0$ randomly, namely, we generate 2 sets of volume $N$ from a standard normal distribution

$$x_0, v_0 \sim N(0, 1).$$

In the above described way we construct a family of the solutions of Eq. (4). We extend model (4), (2), (3) by adding periodical perturbation

$$x'_{new}(t_k) = \alpha x'_{old}(t_k) + v_{rand}(t_k), \quad 0 \leqslant \alpha \leqslant 1, \quad t_k = k\tau, \tag{5}$$

where $v_{rand}(t_k)$, $k = 1, 2, \ldots$ are random quantities. Note, that $v_{rand}(t_k)$ depends on the perturbation moment $t_k$ but not on the element of the family of the solutions. The simplest and the most natural way is to choose sequence $v_{rand}(t_k)$ from a standard normal distribution $N(0, 1)$. Note also, that the case of perturbation period $\tau = \infty$ corresponds to the evolution without perturbation (5), i.e. governed exclusively by Eq. (4).

In Fig. 1 by direct calculations in the phase space $(x, x')$ we observe a transition from the actual chaotic dynamics for large $\tau$ to the nonchaotic common for the whole family of possible solutions evolution with the decrease of $\tau$. For the non perturbed ($\tau = \infty$) family of Eq. (4) solutions there is no transition to the common trajectory, for smaller $\tau = 0.4$ the clustering process of particles with different initial conditions is relatively slow while for sufficiently small $\tau = 0.2$ a collapse to the common trajectory at the time moment $t = 100$ is evident. See Fig. 2 for the generalized result in the terms of Lyapunov exponents with the same parameters values.

## MAPPING EQUATIONS AND LYAPUNOV EXPONENTS

Theoretically a transition from chaotic to nonchaotic behavior can be detected from analysis of two neighboring solutions $x(t)$ and $x^{(1)}(t)$ initially at points $x_0$ and $x_0^{(1)}$ with starting derivatives $v_0$ and $v_0^{(1)}$, respectively. We denote $v = dx/dt$. The convergence of two solutions to the single final evolution depends on the propagation with a time of the small variances $\Delta x = x^{(1)} - x$ and $\Delta v = v^{(1)} - v$. From the

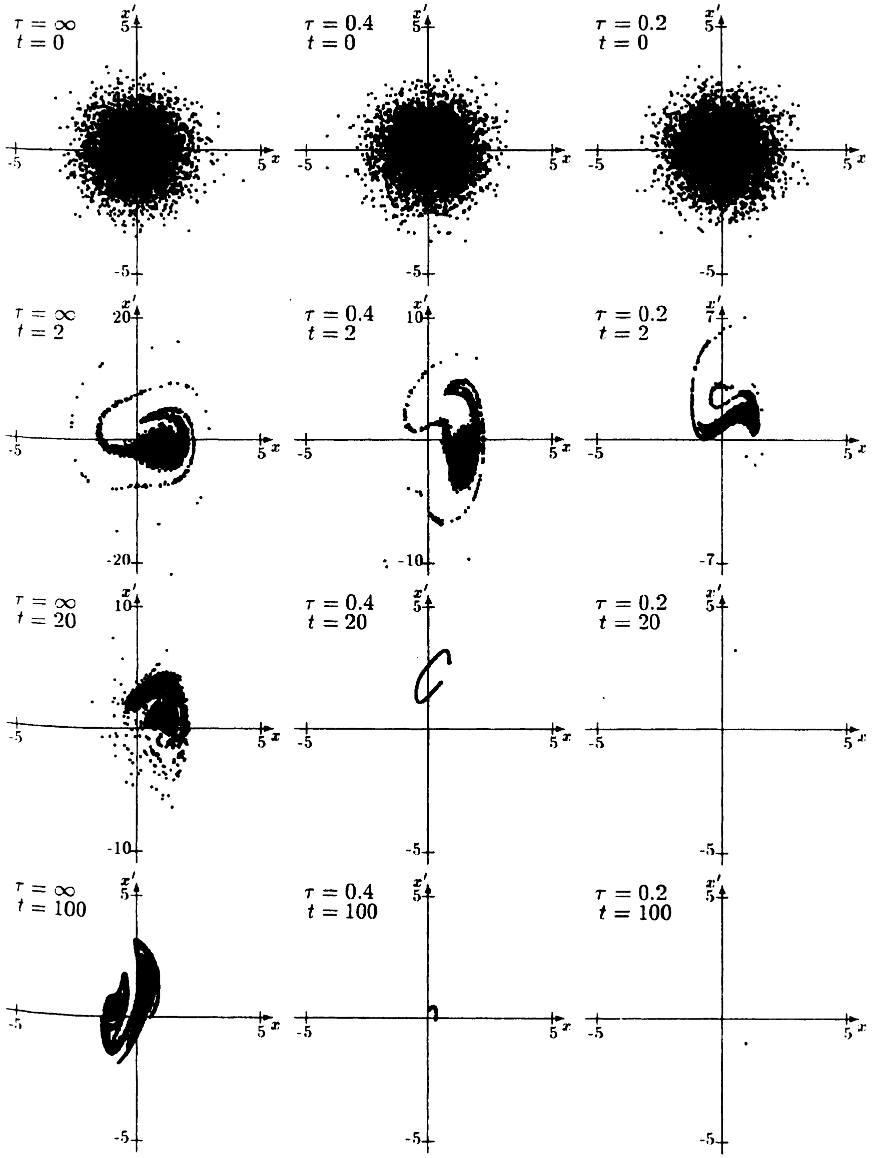


**Figure 1.** Evolution in the phase-space $(x, x')$ in the case $\gamma = 0.07$, $a = 5$, $\alpha = 0.8$ and $N = 5000$.

formal expression $x = x(x_k, v_k, t)$ and $v = v(x_k, v_k, t)$ of the Eq. (4) solution with initial conditions $x(t_k) = x_k$ and $v(t_k) = v_k$, $t_k = k\tau$ we obtain the mapping form of the equations of motion for $\Delta x$ and $\Delta v$:

$$\begin{pmatrix} \Delta x_{k+1} \\ \Delta v_{k+1} \end{pmatrix} = \mathbf{T}(\alpha, \tau, x_k, v_k) \begin{pmatrix} \Delta x_k \\ \Delta v_k \end{pmatrix}, \quad k = 0, 1, \dots, \tag{6}$$

where $\mathbf{T}$ is a matrix of the form

$$\mathbf{T} = \begin{pmatrix} T_{xx} & \alpha T_{xv} \\ T_{vx} & \alpha T_{vv} \end{pmatrix} = \begin{pmatrix} \dfrac{x(x_k, v_k, t)}{x_k} & \alpha\dfrac{x(x_k, v_k, t)}{v_k} \\ \dfrac{v(x_k, v_k, t)}{x_k} & \alpha\dfrac{v(x_k, v_k, t)}{v_k} \end{pmatrix}. \qquad (7)$$

Matrix $\mathbf{T}$ elements $T_{xx}$ and $T_{xv}$ satisfy the equation

$$\frac{d^2T_x}{dt^2} = (2 - 12x^2(t))T_x - \gamma\frac{dT_x}{dt}, \quad t_k < t \leqslant t_{k+1}, \quad t_k = k\tau \qquad (8)$$

and initial conditions

$$T_{xx}|_{t=t_k} = 1, \qquad T_{xv}|_{t=t_k} = 0, \qquad (9)$$

$$\left.\frac{dT_{xx}}{dt}\right|_{t=t_k} = 0, \qquad \left.\frac{dT_{xv}}{dt}\right|_{t=t_k} = 1, \qquad (10)$$

while

$$T_{vx} = \frac{dT_{xx}}{dt}, \qquad T_{vv} = \frac{dT_{xv}}{dt}. \qquad (11)$$

Further analysis is based on the general theory of the dynamics of systems represented as maps. For every step, when (5) perturbation is performed, we calculate both eigenvalues $\mu_k^{(1)}$ and $\mu_k^{(2)}$ of the $\mathbf{T}$ matrix and evaluate the averaged Lyapunov exponent or Kolmogorov–Sinai entropy of the system (4), (2), (3), (5)

$$\lambda = \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n}\frac{1}{\tau}\ln\max\left\{\left|\mu_k^{(1)}\right|, \left|\mu_k^{(2)}\right|\right\}. \qquad (12)$$

A criterion for transition to chaotic behavior is $\lambda = 0$.

In Fig. 2 we see the dependence of the Lyapunov exponents $\lambda$ on perturbation period $\tau$. For the values of parameters corresponding to $\lambda > 0$, the system is chaotic. The negative Lyapunov exponents indicate to the nonchaotic evolution. Note, that for $\alpha < 1$ and sufficiently small $\tau$ the Lyapunov exponent is negative, i.e. intensive

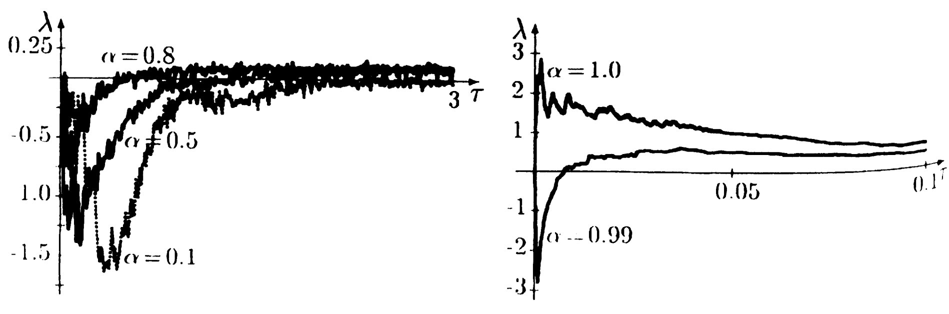


**Figure 2.** Lyapunov exponents $\lambda$ vs the perturbation period $\tau$ for evolution according to Eq. (4) and (5) with $\gamma = 0.07$, $a = 5$ and different $\alpha$.

identical noise (2) synchronizes chaotic system (1), while for $\alpha = 1$ it is positive for all $\tau$.

Comparisons of the threshold values $\tau_c$ for transition to chaos according to Eqs. (6)–(12) with those from the direct numerical simulations (see Fig. 1) indicate to the fitness and usefulness of the method (6)–(12) for investigation of transition from chaotic to nonchaotic behavior in randomly driven ensemble of systems bounded in the fixed external potential with friction and external force.

*Acknowledgment*

The research described in this publication was supported in part by Grant No. 220 of the Lithuanian State Science and Studies Foundation.

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**Identiško triukšmo sinchronizuojanti įtaka chaotinėms sistemoms**

*F. Ivanauskas, T. Meškauskas, B. Kaulakys*

Straipsnyje nagrinėjamas deterministinio chaoso savybe pasižyminčios diferencialinės lygties sprendinių šeimos perėjimas iš chaotinio į nechaotinį elgesį ir sinchronizacija. Antros eilės diferencialinės lygties, su nuo laiko priklausančiu potencialiu bei trinties įskaitymu, sprendiniai periodiškai veikiami vienodo triukšmo. Analizuojama jų sinchronizacijos (nepriklausymo nuo pradinių sąlygų) galimybė. Sistemos chaotiškumas tiriamas įvedus sutiesintas atvaizdžio lygtis ir Liapunovo rodiklius. Nustatomos kritinės sistemos parametrų reikšmės, kurioms stebima sinchronizuojanti triukšmo įtaka chaotinei sistemai bei Liapunovo rodiklių priklausomybė nuo išorinio triukšmo intensyvumo.