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On one signed Poisson approximation

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In the early fifties Kolmogorov proposed to use for approximation the whole class of infinitely divisible distributions. Prokhorov, Kolmogorov, Le Cam, Ibragimov and Presman, Arak, Zaĭtsev and many others devoted their papers to the general and partial solutions of this problem, which became known as the first uniform Kolmogorov theorem. The history of the problem and its solution are in the book Arak and Zaĭtsev (1988). Arak (1981) has proved that any n-fold convolution can be approximated by the infinitely divisible laws with the accuracy $n^{-2/3}$. In Čekanavičius (1997) were discussed possible sets of infinitely divisible measures which can be used as a basis for asymptotic expansions in the first uniform Kolmogorov theorem. In this note we show how one Kornya type expansion in the exponent can be constructed.

We need some notation. Let $\mathcal{F}$ be the set of all one-dimensional distributions, $\mathcal{M}$ be the set of measures of finite variation. $C(\cdot)$ denotes different positive constants depending on the indicated argument only. Let E_a be the distribution concentrated at a point a, $E \equiv E_0$. Products and powers of measures are understood in the convolution sense, $FG = F * G$, $F^0 = E$. For any $W \in \mathcal{M}$ we denote by $supp W$ its support, by $\widehat{W}(t)$ its Fourier–Stieltjes transform, by $\exp\{W\} = \sum_0^\infty W^k/k!$ its exponential measure and by $|W| = \sup_x |W\{(-\infty, x)\}|$ the analogue of the uniform distance. Assume that $W = W^+ - W^-$ is the Jordan–Hahn decomposition of W. Then we denote by $\|W\| = W^+\{\mathbb{R}\} + W^-\{\mathbb{R}\}$ the total variation norm of W. Total variation norm is equivalent to the total variation distance. Let $Q(F, h) = \sup_x F\{[x, x+h]\}$ denote the Levy concentration function.

Definition. The measure $\exp\{\lambda(F - E)\}$, $\lambda \in \mathbb{R}$, $F \in \mathcal{F}$ is called the signed compound Poisson measure.

Note that, if $\lambda \geqslant 0$, we have a compound Poisson distribution but, if $\lambda < 0$, we have a signed measure.

Let us consider a discrete distribution F which is concentrated on $0, x_1, \ldots, x_N$; $p_0 = F\{0\}$, $p_j = F\{x_j\}$, $j = 1, 2, \ldots, N$. Note that F is unnecessary lattice and asymptotics for F^n is almost totaly unexplored. We shall show how to get *infinitely divisible* asymptotic expansion for F^n. Moreover, in the scheme of sequences the rate of accuracy will be $O(n^{-1})$.

Set

$$D = \exp\{(F - E) - (F - E)^2/2 + (\overline{F} - E)^3/3\}. \tag{2}$$

Here $(\overline{F} - E)^3$ corresponds to the following Fourier–Stieltjes transform

$$\sum_{j=1}^{N} p_j^3 \overline{z}_j |z_j|^2 + 3 \sum_{j \neq k} p_j^2 p_k |z_j|^2 \overline{z}_k + \frac{1}{4} \sum_{j \neq k \neq l} p_j p_k p_l (z_j z_k z_l + \overline{z}_j \overline{z}_k z_l + \overline{z}_j z_k \overline{z}_l + z_j \overline{z}_k \overline{z}_l). \tag{3}$$

Here $z_j = \exp\{itx_j\} - 1$, $\overline{z}_j = \exp\{-itx_j\} - 1$.

Now we can formulate the main result of this note.

THEOREM 1. *Let $n \geqslant 1$, $h > 0$. Then*

$$|F^n - D^n| \leqslant C(N) \frac{(1 - p_0)}{n p_0^2} \times \left(1 + \ln(n+1) \min\left\{1, (1 + \max_j |x_j|/h)\Big(n p_0 \sum_{j:|x_j|>h} p_j\Big)^{-1/2}\right\}\right) + e^{-n}. \tag{4}$$

COROLLARY 1. *Assume that all x_j do not depend on n. Then*

$$|F^n - D^n| \leqslant C(N) \frac{(1 - p_0)}{n p_0^2} \left(1 + \frac{\ln(n+1)}{\sqrt{n p_0 (1 - p_0)}}\right) + e^{-n}. \tag{5}$$

COROLLARY 2. *Assume that F does not depend on n. Then*

$$|F^n - D^n| = O(n^{-1}). \tag{6}$$

Proof of Theorem 1. We need some auxiliary results. Set $z = e^{itx} - 1$.

LEMMA 1. *For all $x, t \in \mathbb{R}$, $j \in \mathbb{N}$*

$$z + \overline{z} = -|z|^2, \tag{7}$$

$$z^{4j} = |z|^{4j} + \sum_{l=0}^{2j-1} (-1)^l z^{2l+1} |z|^{4j-2l}, \tag{8}$$

$$z^{4j+1} = z|z|^{4j} + \sum_{l=0}^{2j-1} (-1)^l z^{2l+2} |z|^{4j-2l}, \tag{9}$$

$$z^{4j+2} = -|z|^{4j+2} + \sum_{l=0}^{2j} (-1)^{l+1} z^{2l+1} |z|^{4j+2-2l}, \tag{10}$$

$$z^{4j+3} = \overline{z}|z|^{4j+2} + \sum_{l=0}^{2j+1} (-1)^l z^{2l} |z|^{4j+4-2l}. \tag{11}$$

Proof. Lemma is proved by induction. For example, from (7) and (8) we get

$$zz^{4j+1} = |z|^{4j} + \sum_{l=0}^{2j-1}(-1)^l z^{2l+3}|z|^{4j-2l}$$
$$= z(-\bar{z} - |z|^2)|z|^{4j} + \sum_{l=0}^{2j-1}(-1)^l z^{2(l+1)+1}|z|^{4j+2-2(l+1)}.$$

□

Note that (7)-(11) allow to change z^j by expressions with the non-positive real parts and the remainder is of order $O(|t|^{j+1})$.

LEMMA 2. *Let $G_1, G_2 \in \mathcal{M}$, $G_0 \in \mathcal{F}$, $\widehat{G}_0(t) \geqslant 0$, $G_1\{\mathbb{R}\} = G_2\{\mathbb{R}\}$. Then, for $l, m \in \mathbb{N}$, $u \in \mathbb{R}^l$, $h > 0$, the following inequality holds*

$$|G_1 - G_2| \leqslant C \int_{-1/h}^{1/h} \frac{|\widehat{G}_1(t) - \widehat{G}_2(t)|}{|t|} dt + c(l)\ln(m+1)\sup_t \frac{|\widehat{G}_1(t) - \widehat{G}_2(t)|}{\widehat{G}_0(t)} Q(G_0, h) + \int_{\mathbb{R}\setminus K_m(u)} 1\,(G_1^+ + G_1^-)\{dx\} + \int_{\mathbb{R}\setminus K_m(u)} 1\,(G_2^+ + G_2^-)\{dx\}. \tag{12}$$

Here

$$K_m(u) = \left\{\sum_{i=1}^{l} n_i u_i\colon\ n_i \in \{-m, -m+1, \ldots, m-1, m\},\ i = 1, \ldots, l\right\}. \tag{13}$$

Lemma 2 was proved by Arak (1981) for $G_1, G_2 \in \mathcal{F}$. The proof of (13) is analogous.

LEMMA 3. *Let $W_1, W_2 \in \mathcal{M}$, $n, m \in \mathbb{N}$, $u \in \mathbb{R}^s$, $l \leqslant n$, $j \leqslant n$. Let*

$$\|W_1\| \leqslant a, \qquad \|W_2\| \leqslant b \qquad \textit{and} \quad supp W_i \subset K_m(u),\ i = 1, 2.$$

Then

$$\int_{\mathbb{R}\setminus K_{nm(3a+3b+2)}(u)} 1((W_1^j \exp\{lW_2\})^+ + (W_1^j \exp\{lW_2\})^-)\{dx\} \leqslant e^{-n}. \tag{14}$$

The proof of (14) can be found in Čekanavičius (1992).

LEMMA 4. *Let* $G = \exp\{\alpha(F - E)\}$, $F \in \mathcal{F}$, $\alpha > 0$. *Then, for* $\tau \geqslant 0$,

$$Q(G, \tau) \leqslant C(\alpha F\{x: |x| > \tau\})^{-1/2}. \tag{15}$$

The proof of (15) can be found in Arak and Zaïtsev (1988). Now we can continue the proof of theorem. It is not difficult to check that

$$Re(\widehat{\overline{F}}(t) - 1) < 0. \tag{16}$$

Consequently, by (7)–(11) we get

$$\begin{aligned} |\widehat{F}^n(t) - \widehat{D}^n(t)| &\leqslant n|\widehat{F}(t) - \widehat{D}(t)| \max\left(|F(t)|^{n-1}, |\widehat{D}(t)|^{n-1}\right) \\ &\leqslant n|\widehat{F}(t) - \widehat{D}(t)| \exp\left\{\frac{(n-1)}{2}(|\widehat{F}(t)|^2 - 1)\right\} \\ &\leqslant C \exp\left\{ - p_0(n-1)\sum_{j=1}^N p_j|z_j|^2/2\right\}\left(\left(\sum_{j=1}^N p_j|z_j|^2\right)^2\right. \\ &\quad + \sum_{j=1}^N p_j^3|z_j|^4 + \sum_{j\neq k}^N p_j^2 p_k(|z_j|^3|z_k| + |z_j|^2|z_k|^2) \\ &\quad \left. + \sum_{j\neq k\neq l}^N p_j p_k p_l |z_j|^2|z_k||z_l|\right). \end{aligned} \tag{17}$$

By Hölder's inequality, for all $t \in \mathbb{R}$, we get

$$|\widehat{F}^n(t) - \widehat{D}^n(t)| \leqslant CN^2\frac{(1-p_0)}{np_0^2}\exp\left\{ - \frac{p_0(n-1)}{2}\sum_{j=1}^N p_j \sin^2(tx_j/2)\right\}. \tag{18}$$

For all $|t| \leqslant 1/\max_j |x - j|$, we have $|z_j|^2 \geqslant t^2x_j^2$. Therefore, for $|t| \leqslant 1/\max_j |x_j|$,

$$\begin{aligned} |\widehat{F}^n(t) - \widehat{D}^n(t)| \leqslant Cn\Bigg((1-p_0)&\left(\sum_{j=1}^N p_j x_j^2\right)^2 t^4 \\ &+ |t|(np_0)^{-3/2}\left(\sum_{j=1}^N p_j^{3/2}|x_j| + \sum_{j\neq k}^N p_j p_k^{1/2}(|x_j| + |x_k|)\right. \\ &\left.+ \sum_{j\neq k\neq l}^N (p_j p_k p_l)^{1/2}|x_j|\right)\Bigg) \exp\left\{ - \frac{np_0}{4}\sum_{j=1}^N p_j|z_j|^2\right\}. \end{aligned} \tag{19}$$

We shall apply Lemma 2 with $u = (0, x_1, \ldots, x_N)$, $m = 66n$, $l = N + 1$, $h = \max_j |x_j|$, $G_1 = F^n$, $G_2 = D^n$ and $G_0 = \exp\{(F + F^{(-1)}) - 2E)p_0(n-1)/8\}$, where $\widehat{F^{(-1)}}(t) = \widehat{F}(-t)$. Obviously it suffices to assume $h > 0$. Note that $D^n = \exp\{nW\}$, where $\|W\| \leqslant 20/3$ and $\operatorname{supp} W \subset K_3(u)$. Therefore, by (14) we obtain that the last two summands in the right-hand-side of (12) are less than e^{-n}.

From (19) it follows that

$$\int_{-1/h}^{1/h} \frac{|\widehat{G}_1(t) - \widehat{G}_2(t)|}{|t|} dt \leqslant Cn^{-1}(1 - p_0)p_0^{-2}N^3. \tag{20}$$

From (18) we get

$$\sup_t \frac{|\widehat{G}_1(t) - \widehat{G}_2(t)|}{\widehat{G}_0(t)} \leqslant \frac{C(N)(1 - p_0)}{np_0^2}. \tag{21}$$

To end the proof one should use the following well-known estimate

$$Q(G_0, \tau) \leqslant (1 + (\tau/h))Q(G_0, h), \qquad \tau, h > 0. \tag{22}$$

Proof of the corollaries. It suffices to take $h = \min_j |x_j|/2 > 0$. The case $F \equiv E$ is trivial. □

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Apie vieną ženklą keičiančią Puasono aproksimaciją

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Parodyta kaip konstruoti Kornya tipo asimptotinį skleidinį eksponentėje pirmojoje tolygiojoje Kolmogorovo teoremoje diskrečių negardelinių dydžių sumai.