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On the asymptotics of the nearly non-stationary AR(1) models

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Consider a sequence $(X_{n,k}, 0 \leqslant k \leqslant n)_{n=1}^{\infty}$ of first-order autoregressive processes $AR(1)$ given by

$$\begin{cases} X_{n,0} = 0, \\ X_{n,k} = \beta_n X_{n,k-1} + \varepsilon_k, \qquad k = 1, \dots, n, \end{cases} \tag{1}$$

where $(\varepsilon_k,\ k \geqslant 1)$ are martingale differences with respect to a family of σ-algebras $(\mathcal{F}_k)$ such that $\mathbf{E}\varepsilon_n^2 = 1$ for all n, and β_n is an unknown autoregressive parameter.

If $\beta_n \to 1$ as $n \to \infty$, then model (1) is called a nearly nonstationary (NNS) first-order autoregression.

We investigate the quantity

$$V_n = \left(\sum_{k=1}^{n} X_{n,k-1}^2 \right)^{-1} \sum_{k=1}^{n} \varepsilon_k X_{n,k-1} \tag{2}$$

provided $\sum_{k=1}^{n} X_{n,k-1}^2 \neq 0$.

We further denote $\delta_n = n(1 - \beta_n)$ and assume that

$$0 < 1 - \beta_n \downarrow 0 \quad \text{as} \quad n \to \infty.$$

The aim of this note is to show how the limit distribution of $\tau_n \cdot V_n$, where τ_n is the normalizing constants, depends on the behavior of the quantity δ_n. The obtained results generalize the previous results in [1], [2].

At first we consider NNS model (1) where $\delta_n \to \gamma$, $0 \leqslant \gamma < \infty$.

Denote

$$Z := \left(\int_0^1 Y^2(s) ds \right)^{-1} \int_0^1 Y(s)\, dW(s), \qquad M_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{[nt]} \varepsilon_k,$$

where $W(t)$, $t \in [0, 1]$, is a standard Brownian motion, whereas $Y(t)$, $t \in [0, 1]$, is an Ornstein–Uhlenbeck process defined by the Itô stochastic differential equation

$$dY(t) = -\gamma Y(t)\, dt + dW(t). \tag{3}$$

Recall that for two r.v.'s $\xi, \eta \in \mathbb{R}$ their Ky-Fan distance is defined by

$$\mathcal{K}(\xi, \eta) = \inf_{\varepsilon > 0} \left(\varepsilon + \mathbf{P}\{ |\xi - \eta| \geqslant \varepsilon \} \right)$$

and for $\xi, \eta \in D[0, 1]$

$$\mathcal{K}_\infty(\xi, \eta) = \inf_{\varepsilon > 0} \left(\varepsilon + \mathbf{P}\{ \|\xi - \eta\|_\infty \geqslant \varepsilon \} \right),$$

where $\| \cdot \|_\infty$ is uniform metric.

THEOREM 1. *Let $\delta_n \to \gamma$, $0 \leqslant \gamma < \infty$. Then there exist constants $C_1 = C_1(\gamma)$, C_2, $C_3(\gamma)$ such that*

$$\pi\Big(n \cdot V_n, Z\Big) \leqslant C_1 A_n |\ln A_n|^{7/2} + C_2 \mathcal{K}\Big(\frac{1}{n}\sum_{k=1}^{n} \varepsilon_k^2, 1\Big) |\ln A_n| + C_3 \mathcal{K}_\infty(M_n, W) |\ln A_n|^3,$$

where $A_n = (1-\beta_n) + |\delta_n - \gamma|$.

COROLLARY. *If $\gamma = 0$ then*

$$n \cdot V_n \xrightarrow[n\to\infty]{d} \Big(\int_0^1 W^2(s)\,ds\Big)^{-1} \int_0^1 W(s)\,dW(s).$$

We shall give only a sketch of the proof of Theorem 1.

The following expression easily follows by the definition of an $AR(1)$ process (X_k) and an Ornstein–Uhlenbeck process Y:

$$\begin{aligned} \sum_{k=1}^{n} X_{n,k-1}\varepsilon_k &= \frac{1}{2\beta_n}\Big[X_{n,n}^2 + (1-\beta_n^2)\sum_{k=1}^{n} X_{n,k-1}^2 - \sum_{k-1}^{n}\varepsilon_k^2\Big], \\ \int_0^t Y_s\,dW_s &= \frac{1}{2}Y_t^2 + \gamma\int_0^t Y_s^2\,ds - \frac{1}{2}t. \end{aligned} \tag{4}$$

We have by (3) that

$$\begin{aligned} &\Big| n \cdot V_n - \Big(\int_0^1 Y_s^2\,ds\Big)^{-1}\int_0^1 Y_s\,dW_s\Big| \\ &\leqslant \Big|\frac{n}{2\beta_n}\Big(\sum_{k=1}^{n} X_{n,k-1}^2\Big)^{-1}\Big(X_{n,n}^2 - \sum_{k=1}^{n}\varepsilon_k^2\Big) - \frac{1}{2}(Y_1^2 - 1)\Big(\int_0^1 Y_s^2\,ds\Big)^{-1}\Big| \\ &+ \Big| n\,\frac{1-\beta_n^2}{2\beta_n} - \gamma\Big| = I_1 + I_2. \end{aligned}$$

For any $p, r > 0$ define the set

$$\Omega = \Big\{\sum_{k=1}^{n} X_{k-1}^2 > pn^2\Big\} \cap \Big\{\int_0^1 Y_s^2\,ds > r\Big\} \cap \Big\{|Y_1| \leqslant \sqrt{\lambda}\Big\}.$$

After simple calculations on the set Ω one can get

$$I_1 \leqslant \frac{1}{p\,\beta_n}\left[\left|X_{n,n} - Y_1\right|^2 + 2\sqrt{\lambda}\left|X_n - Y_1\right| + \left|\sum_{k=1}^{n} \varepsilon_{n,k}^2 - 1\right|\right] + \frac{1-\beta_n}{p\,\beta_n}(\lambda+1) + \frac{\lambda+1}{pr}\left[\left\|X^n - Y\right\|_\infty^2 + 2\sqrt{r}\left\|X^n - Y\right\|_\infty\right],$$

where $X_t^n = X_{n,[nt]}$.

Similarly as in [1] we can get such results.

LEMMA 1. *The following estimates are valid*

$$|X_{n,n} - Y_1| \leqslant e^{\delta_n}\left[2|\delta_n - \gamma| \cdot \|W\|_\infty + \|M^n - W\|_\infty\right],$$
$$\|X^n - Y\|_\infty \leqslant 2e^{\delta_n} A_n \cdot \|W\| + e^{\delta_n}\left[(1-\beta_n) + 1\right] \cdot \|M^n - W\|_\infty.$$

LEMMA 2. *If* $A_n < e^{-1}$, *then*

$$\mathcal{K}_\infty(X^n, Y) \leqslant 2\sqrt{2} e^{\delta_n} A_n \left|\ln A_n\right|^{1/2} + e^{\delta_n}\left[2(1-\beta_n) + 1\right]\mathcal{K}_\infty(M^n, W),$$
$$\mathcal{K}\left(\int_0^1 (X_s^n)^2 ds, \int_0^1 Y_s^2 ds\right) \leqslant c_1 \mathcal{K}_\infty(X^n, Y)\left|\ln A_n\right|^{1/2} + c_2 A_n.$$

for some contstants c_1 *and* c_2.

LEMMA 3. *If* $r = c\,|\ln A_n|^{-1}$, *then*

$$\mathbf{P}\left(\int_0^1 Y_s^2 ds \leqslant r\right) \leqslant C\,e^{\gamma/2} A_n,$$

where c *and* C *are some constants.*

LEMMA 4. *If*

$$\mathcal{K}\left(\int_0^1 (X_s^n)^2 ds, \int_0^1 Y_s^2 ds\right) \leqslant c_1\left|\ln A_n\right|^{-1},$$

then

$$\mathbf{P}\left(\int_0^1 (X_s^n)^2 ds \leqslant c_2\left|\ln A_n\right|^{-1}\right) \leqslant \mathcal{K}\left(\int_0^1 (X_s^n)^2 ds, \int_0^1 Y_s^2 ds\right) + C\,e^{\gamma/2} A_n,$$

where c_1, c_2, *and* C *are some constants.*

Now one can finish the proof similarly as in [1].

Now we consider NNS model (1) when $\delta_n \uparrow \infty$. The accuracy of normal approximation of V_n is obtained with respect to the uniform distance

$$\Delta_n = \sup_x \left| \mathbf{P}\left\{ \tau_n V_n < x, \sum_{k=1}^{n} X_{n,k-1}^2 \neq 0 \right\} - \Phi(x) \right|,$$

where $\tau_n = \sqrt{n/(1-\beta_n^2)}$ and $\Phi(x)$ is the standard normal distribution.

Denote by

$$G_n(x) = \mathbf{P}\left(\tau_n^{-1} \sum_{k=1}^{n} X_{k-1}\varepsilon_{nk} < x \right).$$

THEOREM 2. *Let* $\sup_n |\varepsilon_n| \leqslant C$ *a.s. for some constant* C *and* $\delta_n \geqslant 2$, *then there exists a constant* $c > 0$ *such that*

$$\Delta_n \leqslant c\left(\delta_n^{-2/3} \ln \delta_n + \sup_x \left| G_n(x) - \Phi(x) \right| \right).$$

Proof. It is well known that

$$\mathcal{L}(F_n, \Phi) \leqslant \sup_x \left| F_n(x) - \Phi(x) \right| \leqslant \left(1 + \frac{1}{\sqrt{2\pi}} \right) \mathcal{L}(F_n, \Phi),$$

$$\mathcal{L}(F_n, G_n) \leqslant \gamma + \mathbf{P}\left(\left| \tau_n V_n - \tau_n^{-1} \sum_{k=1}^{n} X_{n,k-1}\varepsilon_k \right| > \gamma \right),$$

where $\mathcal{L}(\cdot, \cdot)$ is the Levy distance.

So

$$\begin{aligned} \sup_x \left| F_n(x) - \Phi(x) \right| &\leqslant \left(1 + \frac{1}{\sqrt{2\pi}} \right) \left[\mathcal{L}(F_n, G_n) + \sup_x \left| G_n(x) - \Phi(x) \right| \right] \\ &\leqslant \left(1 + \frac{1}{\sqrt{2\pi}} \right) \Big[\gamma + \mathbf{P}\left(\left| \tau_n V_n \right| \cdot \left| 1 - \tau_n^{-2} T_2 \right| > \gamma \right) \\ &\quad + \sup_x \left| G_n(x) - \Phi(x) \right| \Big] \end{aligned}$$

and

$$\begin{aligned} \mathbf{P}\left(\left| \tau_n V_n \right| \cdot \left| 1 - \tau_n^{-2} T_2 \right| > \gamma \right) &\leqslant \mathbf{P}\left(T_2 < \frac{\mathbf{E} T_2}{2} \right) + \mathbf{P}\left(\tau_n^{-1} \left| \sum_{k=1}^{n} X_{n,k-1}\varepsilon_k \right| > b \right) \\ &\quad + \mathbf{P}\left(2b\tau_n \mathbf{E}^{-1} T_2 \left| 1 - \tau_n^{-2} T_2 \right| > \gamma \right) = \sum_{k=1}^{3} J_k, \end{aligned}$$

where

$$T_2 = \sum_{k=1}^{n} X_{n,k-1}^2.$$

Since $\sup_n |\varepsilon_n| \leqslant C$, then there exists a constant c such that (see [2], lemma 3.2)

$$J_1 \leqslant c\delta_n^{-1}.$$

Now we estimate J_2. From exponential inequality (see [3]) we get

$$\mathbf{P}\left(\tau_n^{-1}\left|\sum_{k=1}^{n} X_{n,k-1}\varepsilon_k\right| > b\right) \leqslant \mathbf{P}\left(\max_{k\leqslant n} \tau_n^{-1}|X_{n,k-1}\varepsilon_k| > u\right) + 2\mathbf{P}(\tau_n^{-2}T_2 > v)$$
$$+ 2\exp\left\{bu^{-1}\left(1 - \ln\left(buv^{-1}\right)\right)\right\}$$

Since $\sup_n |\varepsilon_n| \leqslant C$, then $\max_{1\leqslant k\leqslant n} \tau_n^{-1}|X_{n,k-1}\varepsilon_k| \leqslant \sqrt{2}M^2\delta_n^{-1/2}$, where $M = C \vee 1$. Put

$$b = e^2 \ln^{3/2}\delta_n, \qquad u = 2M^2\ln^{-1/2}\delta_n, \qquad v = 2M^2\ln\delta_n.$$

Then

$$\mathbf{P}\left(\tau_n^{-1}\left|\sum_{k=1}^{n} X_{n,k-1}\varepsilon_k\right| > b\right) \leqslant 2\mathbf{P}(\tau_n^{-2}T_2 > 2M^2\ln\delta_n)$$
$$+ 2\exp\left\{-e^2\left(2M^2\right)^{-1}\delta_n\ln\delta_n\right\}$$
$$\leqslant 2\mathbf{P}\left(|T_2 - \mathbf{E}T_2| > \tau_n^2\right) + c_1\delta_n^{-1} \leqslant c_2\delta_n^{-1}.$$

Now we estimate J_3. For $\delta_n \geqslant 2$ we have $\mathbf{E}T_2 \geqslant \tau_n^2/2$. So

$$J_3 \leqslant \mathbf{P}\left(4b\tau_n^{-3}\left|T_2 - \tau_n^2\right| > \gamma\right)$$
$$\leqslant c_3 b^2\tau_n^{-6}\gamma^{-2}\left(\mathbf{E}|T_2 - \mathbf{E}T_2|^2 + \left(1-\beta_n^2\right)^{-4}\right) \leqslant c_4\delta_n^{-2/3}\ln\delta_n$$

if we set $\gamma = n^{-2/3}\ln\delta_n$.

The proof of the theorem follows from obtained estimates.

COROLLARY. *Suppose that assumptions of Theorem 2 are fulfilled. Then there exists a constant* $c > 0$ *such that*

$$\Delta_n \leqslant c\delta_n^{-1/5}.$$

THEOREM 3. *Let* $\sup_n \mathbf{E}|\varepsilon_n|^p < \infty$ *for some* $p > 2$ *and* $\delta_n \geqslant 2$, *then there exists a constant* $c > 0$ *such that*

$$\Delta_n \leqslant c\left(n^{(4-3p)/(4+3p)}\ln^{2/3}\delta_n + \sup_x\left|G_n(x) - \Phi(x)\right|\right).$$

The proof is just like the proof of Theorem 2.

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Apie beveik nestacionarių $AR(1)$ modelių asimptotiką

K. Kubilius

Tiriama pirmos eilės autoregresinio proceso asimptotinio elgesio priklausomybė nuo parametro elgesio.