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On a securities price binomial model

Remigijus Leipus, Alfredas Račkauskas

1. INTRODUCTION

Assume that the market functioning at the moments $n = 0, 1, \ldots$ contains d securities with positive prices $S_n^1, \ldots, S_n^d$. The securities under consideration may be stocks, zero-coupon bonds with different maturities or other financial assets. By a binomial market of the securities one understands the quadruple

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P}, \{S_n, n = 0, 1, \ldots\}),$$

where $\Omega = \{-1, 1\}^{\mathbb{N}}$, $\mathcal{F}$ is its Borel σ-algebra, $\mathbf{P}$ is the usual invariant measure, $\mathbb{F} = \{\mathcal{F}_n, n = 0, 1, \ldots\}$, where $\mathcal{F}_n = \sigma(\varepsilon_1, \ldots, \varepsilon_n)$ is σ-algebra generated by the first n coordinates of $\varepsilon = \{\varepsilon_1, \varepsilon_2, \ldots\} \in \Omega$ and $\{S_n = (S_n^1, \ldots, S_n^d), n = 0, 1, \ldots\}$ is $\mathbb{F}$-adapted process. A binomial market for stock prices was introduced in the seminal paper of Cox [1] (see also Cox and Rubinstein [2]). In their model the market functioning at the moments $n = 0, 1, \ldots$ contains a non-risky security (bond) and a risky security (stock). The bond prices satisfy the equation

$$S_n^0 = (1 + r)S_{n-1}^0, \quad S_0^0 > 0,$$

where the interest rate $r > 0$, whereas the stock prices are assumed to satisfy the equation

$$S_n^1(\varepsilon) = S_n^1(\varepsilon_1, \ldots, \varepsilon_n) = x(\varepsilon_n)S_{n-1}^1(\varepsilon_1, \ldots, \varepsilon_{n-1}), \quad S_0^1 > 0,$$

where $\varepsilon = \{\varepsilon_n\} \in \Omega$ and $x\colon \{-1; 1\} \to (0, \infty)$ is some positive function satisfying

$$0 < x(-1) < 1 + r < x(1).$$

Defining η_0 so that

$$\eta_0 \, \frac{x(1)}{1 + r} + (1 - \eta_0) \, \frac{x(-1)}{1 + r} = 1$$

we see that $\{S_n^1/S_n^0\}$ is a martingale sequence with respect to the measure $\mathbf{P}_\eta$ and the filtration $\mathbb{F}$, where

$$\mathbf{P}_{\eta_0} = (\eta_0\delta_1 + (1 - \eta_0)\delta_{-1})^{\mathbb{N}}.$$

Thus, by the well-known criteria (see [3, 4]), the market $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P}, \{(S_n^0, S_n^1), n = 0, 1, \ldots\})$ is arbitrage-free and complete, and vice versa: If this market is arbitrage-free, then there exists $\eta \in (0, 1)$ such that $\{S_n^1/S_n^0, n = 0, 1, \ldots\}$ is a martingale sequence with respect to the measure $\mathbf{P}_\eta$ and the filtration $\mathbb{F}$.

Extensions of the binomial model to general discrete time arbitrage-free securities markets were provided later in many works (see, for instance, [4, 5, 7]).

In this paper we consider an estimation of a binomial market model of an arbitrage-free securities market, assuming that there no risk-free investment exists. The role of numeraire is played by the price of some fixed (which remains unchanged during the time moments 0, 1, ...) so-called *benchmark* portfolio of the securities. Such model was introduced in [8].

The second section contains some binomial models of arbitrage-free securities market, supplementing those given in Leipus and Račkauskas (1997). Estimation of parameters involved in these models are investigated in the third section.

2. MODELS

In this section we consider some examples of $\mathbb{F}$-adapted processes $\{S_n, n = 0, 1, \dots\}$ such that $(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P}, \{S_n, n = 0, 1, \dots\})$ provides a binomial arbitrage free market model. Depending on context, for the sake of convenience we use the same notation for the random variable $S_n : \Omega \to (0, \infty)^d$ and for the function $S_n : \{-1, 1\}^n \to (0, \infty)^d$.

Definition 2.1. A set $\{S_0, ; S_n(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n); \varepsilon_i = \pm 1, i = 1, \dots, n; n \in \mathbb{N}\} \subset \mathbb{R}_+^d$ is said to be a tree starting at S_0. Setting for $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots) \in \Omega$,

$$S_n(\varepsilon) = S_n(\varepsilon_1, \dots, \varepsilon_n), \quad n = 1, 2, \dots, \tag{2.1}$$

where $\{S_n, n = 0, 1, \dots\}$ is a tree starting at S_0, we obtain an example of $\mathbb{F}$-adapted process $\{S_n, n = 0, 1, \dots\}$. Therefore, in what follows we discuss the models of a tree $\{S_0; S_n(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n); \varepsilon_i = \pm 1, i = 1, \dots, n; n \in \mathbb{N}\}$. In what follows $\langle \cdot, \cdot \rangle$ denotes the inner product in the d-dimensional vector space $\mathbb{R}^d$: $\langle x, y \rangle = \sum_i x_i y_i$ for $x = (x_1, \dots, x_d)$, $y = (y_1, \dots, y_d)$. Let $\mathbb{R}_+^d = \{a = (a_1, \dots, a_d) \in \mathbb{R}^d : a_1 \geqslant 0, \dots, a_d \geqslant 0\}$. Fix $a \in \mathbb{R}_+^d$, $a = (a_1, \dots, a_d)$, such that $\langle a, a \rangle = 1$. Let $L_a = \{ta, t \in \mathbb{R}\}$ and let $\widetilde{e}_1, \dots, \widetilde{e}_{d-1}$ be an orthonormal basis in the hyperplane

$$L_a^\perp = \{x \in \mathbb{R}^d : \langle x, a \rangle = 0\}.$$

For convenience set $\widetilde{e}_d = a$. For $x, y \in \mathbb{R}^d$ define a vector product

$$x \widetilde{\otimes} y = \sum_{k=1}^{d} \langle x, \widetilde{e}_k \rangle \langle y, \widetilde{e}_k \rangle \widetilde{e}_k.$$

- Consider the tree defined recursively by the equation

$$S_n(\varepsilon_1, \dots, \varepsilon_n) = S_{n-1}(\varepsilon_1, \dots, \varepsilon_{n-1}) \widetilde{\otimes} x(\varepsilon_n), \quad n \in \mathbb{N}, \tag{2.2}$$

where $x\colon \{-1; 1\} \to \mathbb{R}_+^d$, $x(-1) < x(+1)$. Equation (2.2) means that the vector of prices at the moment n is obtained by vector-multiplying the $n-1$-moment prices by random vector x depending on ε_n solely.

Consider the following condition on the values $x(1)$ and $x(-1)$ and the vectors $a, \widetilde{e}_1, \dots, \widetilde{e}_{d-1}$.

CONDITION (F). It holds that

$$\alpha_1 = \dots = \alpha_{d-1} = \alpha \in (0, 1),$$

where

$$\alpha_j = \frac{1 - \frac{\langle x(-1), \widetilde{e}_j \rangle}{\langle x(-1), a \rangle}}{\frac{\langle x(1), \widetilde{e}_j \rangle}{\langle x(1), a \rangle} - \frac{\langle x(-1), \widetilde{e}_j \rangle}{\langle x(-1), a \rangle}} \tag{2.3}$$

for $j = 1, \dots, d-1$.

THEOREM 2.1. *Assume that* $x\colon \{+1; -1\} \to \mathbb{R}^d$, $a, \widetilde{e}_1, \dots, \widetilde{e}_{d-1}$ *satisfies condition* (F). *Then the binomial market*

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P}, \{S_n, n = 0, 1, \dots\}),$$

where S_n *is defined by* (2.2) *is arbitrage free.*

Proof. It is esy to check that $S_n / \langle S_n, a \rangle, n = 0, 1, \dots$ is martingale with respect to the measure

$$\mathbf{P}_\alpha = \big(\alpha \delta_1 + (1 - \alpha) \delta_{-1}\big)^{\mathbb{N}}$$

which is equivalent to the measure $\mathbf{P}$. Indeed, by (2.2) and condition (F)

$$\begin{aligned} \mathbf{E} \frac{S_n}{\langle S_n, a \rangle} \Big| \mathcal{F}_{n-1} &= \frac{S_{n-1}}{\langle S_{n-1}, a \rangle} \widetilde{\otimes} \dot{\mathbf{E}} \frac{x(\varepsilon_n)}{\langle x(\varepsilon_{n-1}), a \rangle} \Big| \mathcal{F}_{n-1} \\ &= \frac{S_{n-1}}{\langle S_{n-1}, a \rangle} \widetilde{\otimes} \left[\alpha \frac{x(1)}{\langle x(1), a \rangle} + (1 - \alpha) \frac{x(-1)}{\langle x(-1), a \rangle} \right] = S_{n-1}. \end{aligned}$$

Now the result follows by Theorem 2.1 from [8].

Remark 2.1. Note that the condition $\alpha_j \in (0, 1)$, where α_j is defined by (2.3) is equivalent to the following one: either

$$\langle x(-1), \widetilde{e}_j \rangle \ll \langle x(-1), a \rangle, \quad < \langle x(1), a \rangle \ll \langle x(1), \widetilde{e}_j \rangle$$

or

$$\langle x(-1), a \rangle \ll \langle x(-1), \widetilde{e}_j \rangle, \quad < \langle x(1), \widetilde{e}_j \rangle \ll \langle x(1), a \rangle.$$

Particulary, consider the case where the function $x = (x_1, \dots, x_d)$ is known. Then we can choose the portfolio a as follows: firstly checking the condition whether there exist k_0, $1 \leqslant k_0 \leqslant d$, and $J \subset \{1, \dots, d\}$, such that

$$\begin{cases} x_j(-1) < x_{k_0}(-1), \ x_{k_0}(1) < x_j(1), & \forall j \in J, j \neq k_0, \\ x_{k_0}(-1) < x_j(-1), \ x_j(1) < x_{k_0}(1), & \forall j \notin J, j \neq k_0. \end{cases} \tag{2.4}$$

If such k_0 exists one puts $a = e_{k_0}$, where $e_1, \dots, e_d$ denotes standard orthonormal basis vectors.

If such k_0 does not exists then one has to change the standard basis to another one, say $\widetilde{e}_1, \ldots, \widetilde{e}_d$ and then to check the condition (2.4) with $x_j = \langle x, \widetilde{e}_j \rangle$. It is easy to see that one can construct the examples where condition (2.4) is not valid with standard basis whereas valid with another basis. When there is no such basis, the arbitrage opportunity exists.

•• Let $q \geqslant 1$ be a fixed number. Consider the model

$$S_n(\varepsilon_1, \ldots, \varepsilon_n) = [S_{n-1}(\varepsilon_1, \ldots, \varepsilon_{n-1}) \widetilde{\otimes} x(\varepsilon_n)] X(\varepsilon_{n-q+1}, \ldots, \varepsilon_n), \tag{2.5}$$

where the function $x: \{-1; 1\} \to \mathbb{R}^d_+$ satisfies

$$\langle x(\pm 1), a \rangle = 1; \tag{2.6}$$

$X: \{+1; -1\}^q \to \mathbb{R}_+$. The factor X is acountable for the price change of the benchmarkt portfolio that can occur accounting the last q steps in the path of the prices states. Indeed, the vector $\{\varepsilon_1, \ldots, \varepsilon_n\}$ describes the hole path by which S_n came to the state number $\varepsilon'_1 + \ldots + \varepsilon'_n$, where

$$\varepsilon'_k = \frac{1 - \varepsilon_k}{2}, \quad k = 1, 2, \ldots$$

The vector $(\varepsilon_{n-q+1}, \ldots, \varepsilon_n)$ describes the last q steps of the hole path. Due to the condition (2.6)

$$\langle S_n, a \rangle = X(\varepsilon_{n-q+1}, \ldots, \varepsilon_n) \langle S_{n-1}, a \rangle.$$

Note that taking into acount only q last steps, only q different states $\varepsilon'_{n-q+1} + \ldots + \varepsilon'_n$ can occur for $\langle S_n, a \rangle$.

Consider the following condition on the values $x(+1)$ and $x(-1)$ and the vectors $a, \widetilde{e}_1, \ldots, \widetilde{e}_{d-1}$.

CONDITION (F1). The orthonormal basis $\{\widetilde{e}_1, \ldots, \widetilde{e}_d\}$ and values $x(-1), x(+1)$ are such that

$$\alpha_1 = \cdots = \alpha_{d-1} = \alpha \in (0, 1),$$

where

$$\alpha_j = \frac{1 - \langle x(-1), \widetilde{e}_j \rangle}{\langle x(1), \widetilde{e}_j \rangle - \langle x(-1), \widetilde{e}_j \rangle} \tag{2.7}$$

for $j = 1, \ldots, d-1$.

THEOREM 2.2. *Assume that* $x : \{+1; -1\} \to \mathbb{R}^d$, $a, \widetilde{e}_1, \ldots, \widetilde{e}_{d-1}$ *satisfies* (2.6) *and condition* $(F1)$. *Then the binomial market*

$$(\Omega, \mathcal{F}, \mathbb{F}, \mathbf{P}, \{S_n, n = 0, 1, \ldots\}),$$

where S_n *is defined by* (2.5) *is arbitrage free.*

Proof. It is esy to check that $S_n / \langle S_n, a \rangle, n = 0, 1, \ldots$ is martingale with respect to the measure $\mathbf{P}_\alpha$.

3. ESTIMATION

• Consider the estimate of the binomial market, where

$$S_n(\varepsilon_1, \ldots, \varepsilon_n) = S_{n-1}(\varepsilon_1, \ldots .\varepsilon_{n-1})\widetilde{\otimes}x(\varepsilon_n). \tag{3.1}$$

Assume that we are given the data $S_0, S_1, \ldots, S_N$, that follows the model (3.1) with certain accuracy. Namely, assume that

$$S_k = e^{\xi_k} S_{k-1}\widetilde{\otimes}x(\varepsilon_n), \quad k = 1, 2, \ldots, N,$$

where $\xi_1, \ldots, \xi_N$ are iid zero mean random variables.

Let $u_j = \langle x(+1), \widetilde{e}_j\rangle$, $d_j = \langle x(-1), \widetilde{e}_j\rangle$, where $j = 1, \ldots, d$. Set

$$\widehat{u}_j = \left(\prod_{k=1}^{N}\left(\frac{\langle S_k, \widetilde{e}_j\rangle}{\langle S_{k-1}, \widetilde{e}_j\rangle} \vee 1\right)\right)^{1/N_j}, \qquad \widehat{d}_j = \left(\prod_{k=1}^{N}\left(\frac{\langle S_k, \widetilde{e}_j\rangle}{\langle S_{k-1}, \widetilde{e}_j\rangle} \wedge 1\right)\right)^{1/(N-N_j)},$$

where

$$N_j = card\{k = 1, \ldots, N : \frac{\langle S_k, \widetilde{e}_j\rangle}{\langle S_{k-1}, \widetilde{e}_j\rangle} > 1\}.$$

Having $\widehat{x}(+1)$ and $\widehat{x}(-1)$ one can check the condition (F) with x replased by $\widehat{x}$. If the condition is satisfied then set

$$\widehat{\alpha} = \frac{1 - \langle\widehat{x}(-1), \widetilde{e}_j\rangle\langle\widehat{x}(-1), a\rangle^{-1}}{\langle\widehat{x}(1), \widetilde{e}_j\rangle\langle\widehat{x}(1), a\rangle^{-1} - \langle\widehat{x}(-1), \widetilde{e}_j\rangle\langle\widehat{x}(-1), a\rangle^{-1}}.$$

Hence, all parameters involved in the model are estimated.

•• Consider the estimate of the binomial market, where

$$S_n(\varepsilon_1, \ldots, \varepsilon_n) = [S_{n-1}(\varepsilon_1, \ldots .\varepsilon_{n-1})\widetilde{\otimes}x(\varepsilon_n)]X(\varepsilon_n). \tag{3.2}$$

Let $u_j = \langle x(+1), \widetilde{e}_j\rangle$, $d_j = \langle x(-1), \widetilde{e}_j\rangle$, where $j = 1, \ldots, d$. Assume that our data $S_0, S_1, \ldots, S_N$ follows

$$S_n = e^{\xi_n} S_{n-1}\widetilde{\otimes}x(\varepsilon_n)X(\varepsilon_n),$$

where as above $\xi_1, \ldots, \xi_N$ are iid zero mean random variables. Set

$$V_k^{(j)} = \frac{\langle S_k, \widetilde{e}_j\rangle\langle S_{k-1}, a\rangle}{\langle S_{k-1}, \widetilde{e}_j\rangle\langle S_k, a\rangle}, \quad k = 1, \ldots, N;$$

and

$$I_1^{(j)} = \{k : V_k^{(j)} > 1\}, \quad I_2 = \{k : V_k^{(j)} < 1\}, \quad \overline{N}_j = card I_1^{(j)}.$$

Now define for $j = 1, \ldots, d-1$

$$\widehat{u}_j = \left(\prod_{k\in I_1} V_k\right)^{1/\overline{N}_j}, \qquad \widehat{d}_j = \left(\prod_{k\in I_2} V_k\right)^{1/(N-\overline{N}_j)}.$$

Next set

$$v_k = \frac{\langle S_k, a\rangle}{\langle S_{k-1}, a\rangle}, \quad k = 1, \ldots, N;$$

and $J_1 = \{k : v_k > 1\}$, $J_2 = \{k : v_k < 1\}$, $M = card J_1$. Now define

$$\widehat{X}(+1) = \Big(\prod_{k\in J_1} v_k\Big)^{1/M}, \quad \widehat{X}(-1) = \Big(\prod_{k\in J_2} v_k\Big)^{1/(N-M)}.$$

Having $\widehat{x}(\pm 1)$ one can check the condition (F1) with x replased by $\widehat{x}$. If the condition is satisfied then set

$$\widehat{\alpha} = \frac{1 - \langle \widehat{x}(-1), \widetilde{e}_j\rangle}{\langle \widehat{x}(+1), \widetilde{e}_j\rangle - \langle \widehat{x}(-1), \widetilde{e}_j\rangle}.$$

Hence, all parameters involved in the model are estimated.

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Binominio medžio privalumai

R. Leipus, A. Račkauskas

Binominio medžio privalumai modeliuojant vertybinių popierių kainas atskleisti Coxo, Roso, Rubinsteino darbuose (gerai žinomi CRR modeliai). Juose, kaip ir daugelyje kitų panašaus pobūdžio darbų, bearbitražės rinkos kriterijai, kaip taisyklė, išreiškiami taikant diskontavimą nerizikingų vertybinių popierių (banko sąskaitos, obligacijos ir pan.) kaina. Toks diskontavimas susilaukia nemažai kritikos. Visų pirma dėl pačių nerizikingų vertybinių popierių egzistavimo. Ypač tai ryšku nagrinėjant besivystančias iš socialistinių į laisvasias rinkas.

Mūsų darbe nagrinėjamas vertybinių popierių kainų modelis, nedarant jokių prielaidų apie nerizikingų vertybinių popierių egzistavimą rinkoje. Bearbitražės rinkos kriterijus išreiškiamas taikant diskontavimą fiksuoto tų pačių vertybinių popierių portfelio kaina. Pateikiami keli binominiai vertybinių popierių kainų modeliai atitinkantys bearbitražę rinką. Aptariamos tų modelių parametrų įvertinimų schemos.