

Positive solutions for a system of fourth-order differential equations with integral boundary conditions and two parameters*

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Abstract. In this work, we investigate a class of nonlinear fourth-order systems with coupled integral boundary conditions and two parameters. We give the Green's functions for the system with boundary conditions, and then obtain some useful properties of the Green's functions. By using the Guo–Krasnosel'skii fixed point theorem and the Green's functions, some sufficient conditions for the existence of positive solutions are presented. As applications, two examples are presented to illustrate the application of our main results.

Keywords: integral boundary conditions, two parameters, Green's function, positive solution, Guo–Krasnosel'skii fixed point theorem.

1 Introduction

The purpose of this paper is to consider the existence of positive solutions for the following system of fourth-order differential equations:

$$\begin{aligned}u^{(4)}(t) &= \lambda f(t, u(t), v(t)), & t \in [0, 1], \\v^{(4)}(t) &= \mu g(t, u(t), v(t)), & t \in [0, 1],\end{aligned}\tag{1}$$

subject to the coupled integral boundary conditions

$$\begin{aligned}u(0) = u'(1) = u'''(1) = 0, & \quad u''(0) = \int_0^1 h_1(s)v''(s) ds, \\v(0) = v'(1) = v'''(1) = 0, & \quad v''(0) = \int_0^1 h_2(s)u''(s) ds,\end{aligned}\tag{2}$$

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where λ and μ are two positive parameters and $h_1, h_2 \in C[0, 1]$. To our knowledge, there has no papers studied system (1) and the coupled integral boundary conditions (2). In this paper, we will establish some sufficient conditions on two parameters λ, μ and nonlinear terms f, g such that positive solutions of (1)–(2) exist. Here positive solutions of (1)–(2) mean a pair of functions $(u, v) \in C[0, 1] \times C[0, 1]$ satisfying (1)–(2) and $u(t) \geq 0, v(t) \geq 0$ for all $t \in [0, 1], (u, v) \neq (0, 0)$.

As we know, fourth-order ordinary differential equations are models for bending or deformation of elastic beams, therefore have important applications in engineering and physical sciences. Recently, fourth-order ordinary differential equations with different types of boundary conditions have been studied by many authors via many methods such as nonlinear alternatives of Leray–Schauder, the fixed point theory, the method of upper and lower solutions, Krasnoselskii fixed point theorem, bifurcation theory, the critical point theory, the shooting method, and fixed point theorems on cones. They can be seen in [2–5, 7, 8, 11, 16–18, 21, 24, 26–30] and the references therein. In [21], the authors considered a fully nonlinear fourth-order equation with integral boundary conditions of type

$$\begin{aligned} x^{(4)}(t) &= f(t, x(t), x'(t), x''(t), x'''(t)), \quad t \in [0, 1], \\ x(0) = x'(1) = x'''(1) &= 0, \quad x''(0) = \int_0^1 g(s)x''(s) ds, \end{aligned} \quad (3)$$

where $f \in C([0, 1] \times [0, +\infty)^2 \times [-\infty, 0], [0, +\infty))$, $g \in C([0, 1], [0, +\infty))$, by using a fixed point theorem of cone expansion and compression of norm type, the existence and nonexistence of concave and monotone positive solutions for problem (3) was obtained. In [11], the authors considered the nonlocal fourth-order boundary value problem with a parameter

$$\begin{aligned} u^{(4)}(t) + B(t)u''(t) &= \lambda f(t, u(t), u''(t)), \quad t \in (0, 1), \\ u(0) = u(1) &= \int_0^1 p(s)u(s) ds, \quad u''(0) = u''(1) = \int_0^1 q(s)u''(s) ds, \end{aligned} \quad (4)$$

where $B \in C[0, 1], \lambda > 0$ is a parameter. By using the Krasnoselskii's fixed point theorem and operator spectral theorem, the existence of positive solutions for problem (4) was given. Recently, there are some papers considered differential systems with coupled boundary conditions, see [6, 13–15, 19, 23, 25] for example. However, boundary value problems composed by systems of fourth-order differential equations are still scarce (see [1, 9, 12, 20, 22, 31] for instance). In [31], the authors considered the existence of positive solutions for fourth-order nonlinear singular semipositone system

$$\begin{aligned} u^{(4)}(t) &= f(t, u(t), v(t), u''(t), v''(t)), \quad t \in [0, 1], \\ v^{(4)}(t) &= g(t, u(t), v(t), u''(t), v''(t)), \quad t \in [0, 1], \\ v(0) = v(1) &= v''(0) = v''(1) = 0, \\ u(0) = u(1) &= u''(0) = u''(1) = 0 \end{aligned}$$

with $f, g \in C((0, 1) \times [0, \infty) \times [0, \infty) \times (\infty, 0] \times (\infty, 0], \mathbb{R})$. The existing results were obtained by approximating the fourth-order system to a second-order singular one and using a fixed point index theorem on cones. In [9], the authors studied the existence of positive solutions for systems of the fourth-order singular semipositone Sturm–Liouville boundary value problems

$$\begin{aligned} u_i^{(4)}(t) &= f_i(t, u_1(t), u_2(t), u_1''(t), u_2''(t)), \quad t \in (0, 1), \\ \alpha_i u_i(0) - \beta_i u_i'(0) &= 0, \quad \nu_i u_i(1) - \delta_i u_i'(1) = 0, \\ \alpha_i u_i''(0) - \beta_i u_i'''(0) &= 0, \quad \nu_i u_i''(1) - \delta_i u_i'''(1) = 0, \quad i = 1, 2, \end{aligned}$$

where $\alpha_i, \nu_i > 0, \beta_i, \delta_i \geq 0, \rho_i = \nu_i \beta_i + \alpha_i \nu_i + \alpha_i \delta_i > 0, f_i \in C((0, 1) \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^- \times \mathbb{R}^-, \mathbb{R})$ with $\mathbb{R} = (-\infty, +\infty), \mathbb{R}^+ = [0, +\infty), \mathbb{R}^- = (-\infty, 0]$, and by applying the fixed point index theorem, some sufficient conditions for positive solutions were established.

Motivated by the works mentioned above, we will study the existence of positive solutions for (1)–(2). But we know, the main difficulty of studying fourth-order differential equations is the calculation of the Green’s function for the problem, and it is more complicated than in the second-order and third-order cases. Therefore, we give the Green’s functions for the fourth-order linear differential equation in Section 2 and then obtain some useful properties for the Green’s functions. In Section 3, we define a proper cone and discuss several properties of the equivalent operator on the cone. By employing Green’s functions and the Guo–Krasnosel’skii fixed point theorem, we establish some sufficient conditions on f, g, λ, μ for the existence of at least one positive solutions of (1)–(2) for appropriately chosen parameters. In Section 4, we present two examples to illustrate the application of our main results.

2 Auxiliary results

We consider the fourth-order coupled system

$$\begin{aligned} u^{(4)}(t) &= x(t), \quad t \in [0, 1], \\ v^{(4)}(t) &= y(t), \quad t \in [0, 1], \end{aligned} \tag{5}$$

with the coupled integral boundary conditions

$$\begin{aligned} u(0) = u'(1) = u'''(1) &= 0, \quad u''(0) = \int_0^1 h_1(s)v''(s) \, ds, \\ v(0) = v'(1) = v'''(1) &= 0, \quad v''(0) = \int_0^1 h_2(s)u''(s) \, ds. \end{aligned} \tag{6}$$

Lemma 1. *If $0 < \int_0^1 h_1(s) ds \cdot \int_0^1 h_2(s) ds < 1$ and $x, y \in C[0, 1]$, then the solution of problem (5)–(6) is given by*

$$\begin{aligned} u(t) &= \int_0^1 G_1(t, s)x(s) ds + \int_0^1 G_2(t, s)y(s) ds, \\ v(t) &= \int_0^1 G_3(t, s)y(s) ds + \int_0^1 G_4(t, s)x(s) ds, \quad t \in [0, 1], \end{aligned} \quad (7)$$

where

$$\begin{aligned} G_1(t, s) &= g_1(t, s) + \frac{2t - t^2}{1 - \Delta} \int_0^1 h_1(s) ds \int_0^1 g_2(\tau, s)h_2(\tau) d\tau, \\ G_2(t, s) &= \frac{2t - t^2}{1 - \Delta} \int_0^1 g_2(\tau, s)h_1(\tau) d\tau, \\ G_3(t, s) &= g_1(t, s) + \frac{2t - t^2}{1 - \Delta} \int_0^1 h_2(s) ds \int_0^1 g_2(\tau, s)h_1(\tau) d\tau, \\ G_4(t, s) &= \frac{2t - t^2}{1 - \Delta} \int_0^1 g_2(\tau, s)h_2(\tau) d\tau \quad \forall t, s \in [0, 1], \end{aligned} \quad (8)$$

and

$$\begin{aligned} g_1(t, s) &= \begin{cases} s(t - \frac{1}{2}t^2) - \frac{1}{6}t^3, & 0 \leq s \leq t \leq 1, \\ t(s - \frac{1}{2}s^2) - \frac{1}{6}s^3, & 0 \leq t \leq s \leq 1, \end{cases} \\ g_2(t, s) &= \begin{cases} s, & 0 \leq s \leq t \leq 1, \\ t, & 0 \leq t \leq s \leq 1, \end{cases} \end{aligned} \quad (9)$$

and $\Delta = \int_0^1 h_1(s) ds \cdot \int_0^1 h_2(s) ds$.

Proof. It is easy to conclude that

$$\begin{aligned} u(t) &= \int_0^1 g_1(t, s)x(s) ds + \left(\frac{1}{2}t^2 - t\right)u''(0), \\ v(t) &= \int_0^1 g_1(t, s)y(s) ds + \left(\frac{1}{2}t^2 - t\right)v''(0), \quad t \in [0, 1], \end{aligned} \quad (10)$$

then we have

$$u''(t) = - \int_0^t sx(s) \, ds - t \int_t^1 x(s) \, ds + u''(0),$$

$$v''(t) = - \int_0^t sy(s) \, ds - t \int_t^1 y(s) \, ds + v''(0).$$

Combining with $u''(0) = \int_0^1 h_1(s)v''(s) \, ds$, one obtains that

$$\begin{aligned} u''(0) &= \int_0^1 h_1(s)v''(s) \, ds \\ &= - \int_0^1 \int_\tau^1 \tau y(\tau)h_1(s) \, d\tau \, ds - \int_0^1 \int_0^\tau sy(\tau)h_1(s) \, d\tau \, ds + v''(0) \int_0^1 h_1(s) \, ds \\ &= - \int_0^1 \left(\int_\tau^1 \tau h_1(s) \, ds + \int_0^\tau sh_1(s) \, ds \right) y(\tau) \, d\tau + v''(0) \int_0^1 h_1(s) \, ds \\ &= - \int_0^1 \left(\int_0^1 g_2(\tau, s)h_1(\tau) \, d\tau \right) y(s) \, ds + v''(0) \int_0^1 h_1(s) \, ds. \end{aligned}$$

By the same method, we get

$$v''(0) = - \int_0^1 \left(\int_0^1 g_2(\tau, s)h_2(\tau) \, d\tau \right) x(s) \, ds + u''(0) \int_0^1 h_2(s) \, ds.$$

Then

$$\begin{aligned} u''(0) &= - \frac{1}{1-\Delta} \left[\int_0^1 \left(\int_0^1 g_2(\tau, s)h_1(\tau) \, d\tau \right) y(s) \, ds \right. \\ &\quad \left. + \int_0^1 h_1(s) \, ds \int_0^1 \left(\int_0^1 g_2(\tau, s)h_2(\tau) \, d\tau \right) x(s) \, ds \right], \\ v''(0) &= - \frac{1}{1-\Delta} \left[\int_0^1 \left(\int_0^1 g_2(\tau, s)h_2(\tau) \, d\tau \right) x(s) \, ds \right. \\ &\quad \left. + \int_0^1 h_2(s) \, ds \int_0^1 \left(\int_0^1 g_2(\tau, s)h_1(\tau) \, d\tau \right) y(s) \, ds \right], \end{aligned}$$

then, by (10), the conclusion is established. □

Lemma 2. *The functions g_1 and g_2 given by (9) have the properties:*

- (i) $g_1, g_2 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^+$ are continuous functions, and $g_1(t, s) \geq 0, g_2(t, s) \geq 0$ for all $(t, s) \in [0, 1] \times [0, 1]$ (see [21]);
- (ii) $g_1(t, s) \leq \tilde{g}_1(s)$ for all $(t, s) \in [0, 1] \times [0, 1]$, where $\tilde{g}_1(s) = s/2 - s^3/6$;
- (iii) for any $\sigma \in (0, 1/2)$, we have $\min_{t \in [\sigma, 1-\sigma]} g_1(t, s) \geq (\sigma/2)\tilde{g}_1(s)$ for all $s \in [0, 1]$.

Proof. (ii) It is easy to get the conclusion, then we omit it.

(iii) There are two cases to consider:

If $0 \leq t \leq s \leq 1$, we have $s - 2s^2/3 \geq s/2 - s^3/6$, then

$$\begin{aligned} g_1(t, s) &= t \left(s - \frac{1}{2}s^2 \right) - \frac{1}{6}s^3 \geq t \left(s - \frac{1}{2}s^2 \right) - \frac{1}{6}ts^2 \\ &= t \left(s - \frac{2}{3}s^2 \right) \geq t \left(\frac{1}{2}s - \frac{1}{6}s^3 \right). \end{aligned}$$

If $0 \leq s \leq t \leq 1$, we have $s - s^2/3 \geq s/2 - s^3/6$, then

$$\begin{aligned} g_1(t, s) &= s \left(t - \frac{1}{2}t^2 \right) - \frac{1}{6}t^3 \geq st \left(1 - \frac{1}{2}t \right) - \frac{1}{6}ts^2 \\ &= t \left[s \left(1 - \frac{1}{2}t \right) - \frac{1}{6}s^2 \right] \geq \frac{1}{2}t \left(s - \frac{1}{3}s^2 \right) \geq \frac{1}{2}t \left(\frac{1}{2}s - \frac{1}{6}s^3 \right). \end{aligned}$$

Let $\rho(t) = \min\{t, t/2\}$, then for $\sigma \in (0, 1/2)$, we have

$$\min_{t \in [\sigma, 1-\sigma]} g_1(t, s) \geq \min_{t \in [\sigma, 1-\sigma]} \rho(t)\tilde{g}_1(s) = \frac{\sigma}{2}\tilde{g}_1(s). \quad \square$$

Lemma 3. *If $0 < \Delta = \int_0^1 h_1(s) ds \cdot \int_0^1 h_2(s) ds < 1$, then G_i ($i = 1, 2, 3, 4$) given by (8) are continuous functions on $[0, 1] \times [0, 1]$ and satisfy $G_i(t, s) \geq 0$ for all $(t, s) \in [0, 1] \times [0, 1]$, $i = 1, 2, 3, 4$. Moreover, if $x, y \in C[0, 1]$ satisfying $x(t) \geq 0, y(t) \geq 0$ for all $t \in [0, 1]$, then the unique solution (u, v) of problem (5)–(6) satisfies $u(t) \geq 0, v(t) \geq 0$ for all $t \in [0, 1]$.*

Proof. By the assumptions of this lemma and (i) in Lemma 2 we obtain $G_i(t, s) \geq 0$ for all $(t, s) \in [0, 1] \times [0, 1]$, combining with $x(t) \geq 0, y(t) \geq 0$, then $u(t) \geq 0, v(t) \geq 0$. \square

Lemma 4. *Assume that $0 < \Delta = \int_0^1 h_1(s) ds \cdot \int_0^1 h_2(s) ds < 1$, then the functions G_i ($i = 1, 2, 3, 4$) satisfy the inequalities*

- (i) $G_1(t, s) \leq \tilde{G}_1(s)$ for all $(t, s) \in [0, 1] \times [0, 1]$, where

$$\tilde{G}_1(s) = \tilde{g}_1(s) + \frac{1}{1-\Delta} \int_0^1 h_1(s) ds \int_0^1 g_2(\tau, s)h_2(\tau) d\tau;$$

(i') for $\sigma \in (0, 1/2)$, we have

$$\min_{t \in [\sigma, 1-\sigma]} G_1(t, s) \geq \frac{\sigma}{2} \tilde{G}_1(s) \geq \frac{\sigma}{2} G_1(t', s) \quad \forall (t', s) \in [0, 1] \times [0, 1];$$

(ii) $G_2(t, s) \leq \tilde{G}_2(s)$ for all $(t, s) \in [0, 1] \times [0, 1]$, where

$$\tilde{G}_2(s) = \frac{1}{1-\Delta} \int_0^1 g_2(\tau, s) h_1(\tau) \, d\tau;$$

(ii') for $\sigma \in (0, 1/2)$, we have

$$\min_{t \in [\sigma, 1-\sigma]} G_2(t, s) \geq \frac{\sigma}{2} \tilde{G}_2(s) \geq \frac{\sigma}{2} G_2(t', s) \quad \forall (t', s) \in [0, 1] \times [0, 1];$$

(iii) $G_3(t, s) \leq \tilde{G}_3(s)$ for all $(t, s) \in [0, 1] \times [0, 1]$, where

$$\tilde{G}_3(s) = \tilde{g}_1(s) + \frac{1}{1-\Delta} \int_0^1 h_2(s) \, ds \int_0^1 g_2(\tau, s) h_1(\tau) \, d\tau;$$

(iii') for $\sigma \in (0, 1/2)$, we have

$$\min_{t \in [\sigma, 1-\sigma]} G_3(t, s) \geq \frac{\sigma}{2} \tilde{G}_3(s) \geq \frac{\sigma}{2} G_3(t', s) \quad \forall (t', s) \in [0, 1] \times [0, 1];$$

(iv) $G_4(t, s) \leq \tilde{G}_4(s)$ for all $(t, s) \in [0, 1] \times [0, 1]$, where

$$\tilde{G}_4(s) = \frac{1}{1-\Delta} \int_0^1 g_2(\tau, s) h_2(\tau) \, d\tau;$$

(iv') for $\sigma \in (0, 1/2)$, we have

$$\min_{t \in [\sigma, 1-\sigma]} G_4(t, s) \geq \frac{\sigma}{2} \tilde{G}_4(s) \geq \frac{\sigma}{2} G_4(t', s) \quad \forall (t', s) \in [0, 1] \times [0, 1];$$

Proof. Inequalities (i)–(iv) are evident. Next, we prove the other inequalities. For $\sigma \in (0, 1/2)$, $t \in [\sigma, 1 - \sigma]$, and $t', s \in [0, 1]$, from Lemma 2 we deduce

$$\begin{aligned} G_1(t, s) &= g_1(t, s) + \frac{2t - t^2}{1 - \Delta} \int_0^1 h_1(s) \, ds \int_0^1 g_2(\tau, s) h_2(\tau) \, d\tau \\ &\geq \frac{\sigma}{2} \tilde{g}_1(s) + \frac{2\sigma - \sigma^2}{1 - \Delta} \int_0^1 h_1(s) \, ds \int_0^1 g_2(\tau, s) h_2(\tau) \, d\tau \\ &\geq \frac{\sigma}{2} \left[\tilde{g}_1(s) + \frac{1}{1 - \Delta} \int_0^1 h_1(s) \, ds \int_0^1 g_2(\tau, s) h_2(\tau) \, d\tau \right] \\ &= \frac{\sigma}{2} \tilde{G}_1(s) \geq \frac{\sigma}{2} G_1(t', s), \end{aligned}$$

$$\begin{aligned}
G_2(t, s) &= \frac{2t - t^2}{1 - \Delta} \int_0^1 g_2(\tau, s) h_1(\tau) \, d\tau \geq \frac{2\sigma - \sigma^2}{1 - \Delta} \int_0^1 g_2(\tau, s) h_1(\tau) \, d\tau \\
&\geq \frac{\sigma}{2} \frac{1}{1 - \Delta} \int_0^1 g_2(\tau, s) h_1(\tau) \, d\tau = \frac{\sigma}{2} \tilde{G}_2(s) \geq \frac{\sigma}{2} G_2(t', s).
\end{aligned}$$

By the same method we get (iii') and (iv'). \square

Lemma 5. Suppose $0 < \Delta = \int_0^1 h_1(s) \, ds \cdot \int_0^1 h_2(s) \, ds < 1$, $\sigma \in (0, 1/2)$, $x, y \in C[0, 1]$, and $x(t) \geq 0$, $y(t) \geq 0$ for all $t \in [0, 1]$. Then the solution $(u(t), v(t))$, $t \in [0, 1]$, of problem (5)–(6) satisfies the inequalities

$$\min_{t \in [\sigma, 1 - \sigma]} u(t) \geq \frac{\sigma}{2} \max_{t' \in [0, 1]} u(t'), \quad \min_{t \in [\sigma, 1 - \sigma]} v(t) \geq \frac{\sigma}{2} \max_{t' \in [0, 1]} v(t').$$

Proof. For $\sigma \in (0, 1/2)$, $t \in [\sigma, 1 - \sigma]$, and $t' \in [0, 1]$, from Lemma 4 we deduce

$$\begin{aligned}
u(t) &= \int_0^1 G_1(t, s) x(s) \, ds + \int_0^1 G_2(t, s) y(s) \, ds \\
&\geq \frac{\sigma}{2} \left(\int_0^1 \tilde{G}_1(s) x(s) \, ds + \int_0^1 \tilde{G}_2(s) y(s) \, ds \right) \\
&\geq \frac{\sigma}{2} \left(\int_0^1 G_1(t', s) x(s) \, ds + \int_0^1 G_2(t', s) y(s) \, ds \right) \\
&= \frac{\sigma}{2} u(t'), \\
v(t) &= \int_0^1 G_3(t, s) y(s) \, ds + \int_0^1 G_4(t, s) x(s) \, ds \\
&\geq \frac{\sigma}{2} \left(\int_0^1 \tilde{G}_3(s) x(s) \, ds + \int_0^1 \tilde{G}_4(s) y(s) \, ds \right) \\
&\geq \frac{\sigma}{2} \left(\int_0^1 G_3(t', s) x(s) \, ds + \int_0^1 G_4(t', s) y(s) \, ds \right) \\
&= \frac{\sigma}{2} v(t').
\end{aligned}$$

Then we obtain the conclusion of this lemma. \square

Theorem 1. (See [10]). *Let X be a Banach space, and let $C \subset X$ be a cone in X . Assume that Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \Omega \subset \Omega_2$, and let $A : C \cap (\bar{\Omega}_2 \setminus \Omega_1) \rightarrow C$ be a completely continuous operator such that either*

- (i) $\|Au\| \leq \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|Au\| \geq \|u\|$, $u \in C \cap \partial\Omega_2$, or
- (ii) $\|Au\| \geq \|u\|$, $u \in C \cap \partial\Omega_1$, and $\|Au\| \leq \|u\|$, $u \in C \cap \partial\Omega_2$.

Then A has a fixed point in $C \cap (\bar{\Omega}_2 \setminus \Omega_1)$.

3 Main results

In this section, we will give sufficient conditions on λ , μ , f , and g such that positive solutions with respect to a cone for our problem (1)–(2) exist. We first present the assumptions, which we will use in the sequel:

- (H1) Functions $f, g : [0, 1] \times [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ are continuous.
- (H2) $0 < \Delta = \int_0^1 h_1(s) ds \cdot \int_0^1 h_2(s) ds < 1$.

By using the functions G_i ($i = 1, 2, 3, 4$) from Lemma 4, our problem (1)–(2) can be written equivalently as the following nonlinear system of integral equations:

$$\begin{aligned}
 u(t) &= \lambda \int_0^1 G_1(t, s) f(s, u(s), v(s)) ds + \mu \int_0^1 G_2(t, s) g(s, u(s), v(s)) ds, \quad t \in [0, 1], \\
 v(t) &= \mu \int_0^1 G_3(t, s) g(s, u(s), v(s)) ds + \lambda \int_0^1 G_4(t, s) f(s, u(s), v(s)) ds, \quad t \in [0, 1].
 \end{aligned}$$

We consider the Banach space $X = C[0, 1]$ with supremum norm $\|\cdot\|$ and the Banach space $Y = X \times X$ with the norm $\|(u, v)\|_Y = \|u\| + \|v\|$. We define the cone $P \subset Y$ by $P = \{(u, v) \in Y : u(t) \geq 0, v(t) \geq 0 \forall t \in [0, 1] \text{ and } \min_{t \in [\sigma, 1-\sigma]} (u(t) + v(t)) \geq (\sigma/2)\|(u, v)\|_Y\}$. For $\lambda, \mu > 0$, we introduce the operators $T_1, T_2 : Y \rightarrow X$ and $Q : Y \rightarrow Y$ defined by

$$\begin{aligned}
 T_1(u, v)(t) &= \lambda \int_0^1 G_1(t, s) f(s, u(s), v(s)) ds + \mu \int_0^1 G_2(t, s) g(s, u(s), v(s)) ds, \\
 T_2(u, v)(t) &= \mu \int_0^1 G_3(t, s) g(s, u(s), v(s)) ds + \lambda \int_0^1 G_4(t, s) f(s, u(s), v(s)) ds,
 \end{aligned}$$

and $Q(u, v) = (T_1(u, v), T_2(u, v))$, $(u, v) \in Y$, for $t \in [0, 1]$. By Lemma 1 the positive solutions of our problem (1)–(2) are fixed points of the operator Q .

Lemma 6. Assume that (H1), (H2) hold, and $\sigma \in (0, 1/2)$, then $Q : P \rightarrow P$ is a completely continuous operator.

Proof. Let $(u, v) \in P$ be an arbitrary element. Because $T_1(u, v)$ and $T_2(u, v)$ satisfy problem (1)–(2) for $x(t) = \lambda f(t, u(t), v(t))$, $t \in [0, 1]$, and $y(t) = \mu g(t, u(t), v(t))$, $t \in [0, 1]$, then by Lemma 5 we obtain

$$\begin{aligned} \min_{t \in [\sigma, 1-\sigma]} T_1(u, v)(t) &\geq \frac{\sigma}{2} \max_{t' \in [0, 1]} T_1(u, v)(t') = \frac{\sigma}{2} \|T_1(u, v)\|, \\ \min_{t \in [\sigma, 1-\sigma]} T_2(u, v)(t) &\geq \frac{\sigma}{2} \max_{t' \in [0, 1]} T_2(u, v)(t') = \frac{\sigma}{2} \|T_2(u, v)\|. \end{aligned}$$

Hence, we conclude

$$\begin{aligned} &\min_{t \in [\sigma, 1-\sigma]} [T_1(u, v)(t) + T_2(u, v)(t)] \\ &\geq \min_{t \in [\sigma, 1-\sigma]} T_1(u, v)(t) + \min_{t \in [\sigma, 1-\sigma]} T_2(u, v)(t) \\ &\geq \frac{\sigma}{2} \|T_1(u, v)\| + \frac{\sigma}{2} \|T_2(u, v)\| = \frac{\sigma}{2} \|Q(u, v)\|_Y. \end{aligned}$$

Combining Lemma 3 with (H1) and (H2), we obtain $T_1(u, v)(t) \geq 0$, $T_2(u, v)(t) \geq 0$ for all $t \in [0, 1]$, then we get $Q(u, v) \in P$. Hence, $Q(P) \subset P$. By using standard arguments, we can easily show that T_1 and T_2 are completely continuous, and then Q is a completely continuous operator. \square

For $\sigma \in (0, 1/2)$, we denote by

$$\begin{aligned} A &= \int_0^1 \tilde{G}_1(s) ds, & B &= \int_0^1 \tilde{G}_2(s) ds, & C &= \int_0^1 \tilde{G}_3(s) ds, & D &= \int_0^1 \tilde{G}_4(s) ds, \\ A_\sigma &= \int_\sigma^{1-\sigma} \tilde{G}_1(s) ds, & B_\sigma &= \int_\sigma^{1-\sigma} \tilde{G}_2(s) ds, & C_\sigma &= \int_\sigma^{1-\sigma} \tilde{G}_3(s) ds, & D_\sigma &= \int_\sigma^{1-\sigma} \tilde{G}_4(s) ds, \end{aligned}$$

where $\tilde{G}_1(s)$, $\tilde{G}_2(s)$, $\tilde{G}_3(s)$, and $\tilde{G}_4(s)$ are defined in Lemma 4. We also introduce the extreme limits below

$$\begin{aligned} f_0 &= \lim_{u+v \rightarrow 0^+} \max_{t \in [0, 1]} \frac{f(t, u, v)}{u+v}, & g_0 &= \lim_{u+v \rightarrow 0^+} \max_{t \in [0, 1]} \frac{g(t, u, v)}{u+v}; \\ f_\infty^\sigma &= \lim_{u+v \rightarrow \infty} \min_{t \in [\sigma, 1-\sigma]} \frac{f(t, u, v)}{u+v}, & g_\infty^\sigma &= \lim_{u+v \rightarrow \infty} \min_{t \in [\sigma, 1-\sigma]} \frac{g(t, u, v)}{u+v}. \end{aligned}$$

In the following, we give our main results.

Theorem 2. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$, $f_0, f_\infty^\sigma, g_0, g_\infty^\sigma \in (0, \infty)$, $\alpha_1, \alpha_2 \in [0, 1]$, $\alpha_3, \alpha_4 \in (0, 1)$, $a \in [0, 1]$, $b \in (0, 1)$, $L_1 < L_2$, and $L_3 < L_4$. Then

for each $\lambda \in (L_1, L_2)$ and $\mu \in (L_3, L_4)$, there exists a positive solution $(u(t), v(t))$, $t \in [0, 1]$, for (1)–(2), where

$$L_1 = \max \left\{ \frac{4a\alpha_1}{\sigma^2 f_\infty^\sigma A_\sigma}, \frac{4(1-a)\alpha_2}{\sigma^2 f_\infty^\sigma D_\sigma} \right\}, \quad L_2 = \min \left\{ \frac{b\alpha_3}{f_0 A}, \frac{(1-b)\alpha_4}{f_0 D} \right\},$$

$$L_3 = \max \left\{ \frac{4a(1-\alpha_1)}{\sigma^2 g_\infty^\sigma B_\sigma}, \frac{4(1-a)(1-\alpha_2)}{\sigma^2 g_\infty^\sigma C_\sigma} \right\}, \quad L_4 = \min \left\{ \frac{b(1-\alpha_3)}{g_0 B}, \frac{(1-b)(1-\alpha_4)}{g_0 C} \right\}.$$

Proof. For σ given in theorem, we consider the above cone $P \subset Y$ and the operators T_1 , T_2 , and Q . Let $\lambda \in (L_1, L_2)$ and $\mu \in (L_3, L_4)$, and let $\varepsilon > 0$ be a positive number such that $\varepsilon < f_\infty^\sigma, \varepsilon < g_\infty^\sigma$, and

$$\begin{aligned} \frac{4a\alpha_1}{\sigma^2(f_\infty^\sigma - \varepsilon)A_\sigma} &\leq \lambda, & \frac{4a(1-\alpha_1)}{\sigma^2(g_\infty^\sigma - \varepsilon)B_\sigma} &\leq \mu, \\ \frac{4(1-a)\alpha_2}{\sigma^2(f_\infty^\sigma - \varepsilon)D_\sigma} &\leq \lambda, & \frac{4(1-a)(1-\alpha_2)}{\sigma^2(g_\infty^\sigma - \varepsilon)C_\sigma} &\leq \mu, \\ \frac{b\alpha_3}{(f_0 + \varepsilon)A} &\geq \lambda, & \frac{b(1-\alpha_3)}{(g_0 + \varepsilon)B} &\geq \mu, \\ \frac{(1-b)\alpha_4}{(f_0 + \varepsilon)D} &\geq \lambda, & \frac{(1-b)(1-\alpha_4)}{(g_0 + \varepsilon)C} &\geq \mu. \end{aligned}$$

By using (H2) and definitions of f_0 and g_0 , we deduce that there exists $R_1 > 0$ such that for all $t \in [0, 1]$, $u, v \in \mathbb{R}^+$ with $0 \leq u + v \leq R_1$, we have $f(t, u, v) \leq (f_0 + \varepsilon)(u + v)$ and $g(t, u, v) \leq (g_0 + \varepsilon)(u + v)$. We define the set $\Omega_1 = \{(u, v) \in Y : \|(u, v)\|_Y < R_1\}$. Now let $(u, v) \in P \cap \partial\Omega_1$, that is, $(u, v) \in P$ with $\|(u, v)\|_Y = R_1$ or equivalently $\|u\| + \|v\| = R_1$. Then $u(t) + v(t) \leq R_1$ for all $t \in [0, 1]$, and by Lemma 4 we obtain

$$\begin{aligned} &T_1(u, v)(t) \\ &= \lambda \int_0^1 G_1(t, s) f(s, u(s), v(s)) \, ds + \mu \int_0^1 G_2(t, s) g(s, u(s), v(s)) \, ds \\ &\leq \lambda \int_0^1 \tilde{G}_1(s) f(s, u(s), v(s)) \, ds + \mu \int_0^1 \tilde{G}_2(s) g(s, u(s), v(s)) \, ds \\ &\leq \lambda \int_0^1 \tilde{G}_1(s) (f_0 + \varepsilon)(u(s) + v(s)) \, ds + \mu \int_0^1 \tilde{G}_2(s) (g_0 + \varepsilon)(u(s) + v(s)) \, ds \\ &\leq \lambda (f_0 + \varepsilon) \int_0^1 \tilde{G}_1(s) (\|u\| + \|v\|) \, ds + \mu (g_0 + \varepsilon) \int_0^1 \tilde{G}_2(s) (\|u\| + \|v\|) \, ds \\ &= [\lambda (f_0 + \varepsilon) A + \mu (g_0 + \varepsilon) B] \|(u, v)\|_Y \\ &\leq [b\alpha_3 + b(1 - \alpha_3)] \|(u, v)\|_Y = b \|(u, v)\|_Y, \quad t \in [0, 1]. \end{aligned}$$

Therefore,

$$\|T_1(u, v)\| \leq b\|(u, v)\|_Y, \quad (u, v) \in P \cap \partial\Omega_1.$$

By the similar method we obtain

$$\begin{aligned} T_2(u, v)(t) &= \mu \int_0^1 G_3(t, s)g(s, u(s), v(s)) \, ds + \lambda \int_0^1 G_4(t, s)f(s, u(s), v(s)) \, ds \\ &\leq \mu \int_0^1 \tilde{G}_3(s)g(s, u(s), v(s)) \, ds + \lambda \int_0^1 \tilde{G}_4(s)f(s, u(s), v(s)) \, ds \\ &\leq \mu \int_0^1 \tilde{G}_3(s)(g_0 + \varepsilon)(u(s) + v(s)) \, ds + \lambda \int_0^1 \tilde{G}_4(s)(f_0 + \varepsilon)(u(s) + v(s)) \, ds \\ &\leq \mu(g_0 + \varepsilon) \int_0^1 \tilde{G}_3(s)(\|u\| + \|v\|) \, ds + \lambda(f_0 + \varepsilon) \int_0^1 \tilde{G}_4(s)(\|u\| + \|v\|) \, ds \\ &= [\mu(g_0 + \varepsilon)C + \lambda(f_0 + \varepsilon)D]\|(u, v)\|_Y \\ &\leq [(1-b)\alpha_4 + (1-b)(1-\alpha_4)]\|(u, v)\|_Y = (1-b)\|(u, v)\|_Y, \quad t \in [0, 1]. \end{aligned}$$

Therefore,

$$\|T_2(u, v)\| \leq (1-b)\|(u, v)\|_Y, \quad (u, v) \in P \cap \partial\Omega_1.$$

Further, for $(u, v) \in P \cap \partial\Omega_1$, we deduce

$$\begin{aligned} \|Q(u, v)\|_Y &= \|T_1(u, v)\| + \|T_2(u, v)\| \leq b\|(u, v)\|_Y + (1-b)\|(u, v)\|_Y \\ &= \|(u, v)\|_Y. \end{aligned}$$

Next, by the definitions of f_∞^σ and g_∞^σ , there exists $R'_2 > 0$ such that $f(t, u, v) \geq (f_\infty^\sigma - \varepsilon)(u + v)$ and $g(t, u, v) \geq (g_\infty^\sigma - \varepsilon)(u + v)$ for all $u, v \geq 0$ with $u + v \geq R'_2$ and $t \in [\sigma, 1 - \sigma]$. We take $R_2 = \max\{2R_1, 2R'_2/\sigma\}$ and define $\Omega_2 = \{(u, v) \in Y: \|(u, v)\|_Y < R_2\}$. Then for $(u, v) \in P$ with $\|(u, v)\|_Y = R_2$, one obtains

$$u(t) + v(t) \geq \inf_{t \in [\sigma, 1-\sigma]} (u(t) + v(t)) \geq \frac{\sigma}{2}\|(u, v)\|_Y = \frac{\sigma}{2}R_2 \geq R'_2 \quad \forall t \in [\sigma, 1 - \sigma].$$

Then, by Lemma 4, we conclude

$$\begin{aligned} T_1(u, v)(\sigma) &\geq \lambda \frac{\sigma}{2} \int_0^1 \tilde{G}_1(s)f(s, u(s), v(s)) \, ds + \mu \frac{\sigma}{2} \int_0^1 \tilde{G}_2(s)g(s, u(s), v(s)) \, ds \\ &\geq \lambda \frac{\sigma}{2} \int_\sigma^{1-\sigma} \tilde{G}_1(s)f(s, u(s), v(s)) \, ds + \mu \frac{\sigma}{2} \int_\sigma^{1-\sigma} \tilde{G}_2(s)g(s, u(s), v(s)) \, ds \end{aligned}$$

$$\begin{aligned}
 &\geq \lambda \frac{\sigma}{2} \int_{\sigma}^{1-\sigma} \tilde{G}_1(s)(f_{\infty}^{\sigma} - \varepsilon)(u(s)+v(s)) \, ds + \mu \frac{\sigma}{2} \int_{\sigma}^{1-\sigma} \tilde{G}_2(s)(g_{\infty}^{\sigma} - \varepsilon)(u(s)+v(s)) \, ds \\
 &\geq \lambda \frac{\sigma^2}{4} (f_{\infty}^{\sigma} - \varepsilon) \int_{\sigma}^{1-\sigma} \tilde{G}_1(s) \|(u, v)\|_Y \, ds + \mu \frac{\sigma^2}{4} (g_{\infty}^{\sigma} - \varepsilon) \int_{\sigma}^{1-\sigma} \tilde{G}_2(s) \|(u, v)\|_Y \, ds \\
 &= \left[\lambda \frac{\sigma^2}{4} (f_{\infty}^{\sigma} - \varepsilon) A_{\sigma} + \mu \frac{\sigma^2}{4} (g_{\infty}^{\sigma} - \varepsilon) B_{\sigma} \right] \|(u, v)\|_Y \\
 &\geq [a\alpha_1 + a(1 - \alpha_1)] \|(u, v)\|_Y = a \|(u, v)\|_Y, \quad t \in [0, 1].
 \end{aligned}$$

Therefore,

$$\|T_1(u, v)\| \geq T_1(u, v)(\sigma) \geq a \|(u, v)\|_Y, \quad (u, v) \in P \cap \partial\Omega_2.$$

In a similar method, we deduce

$$\begin{aligned}
 &T_2(u, v)(\sigma) \\
 &\geq \mu \frac{\sigma}{2} \int_0^1 \tilde{G}_3(s)g(s, u(s), v(s)) \, ds + \lambda \frac{\sigma}{2} \int_0^1 \tilde{G}_4(s)f(s, u(s), v(s)) \, ds \\
 &\geq \mu \frac{\sigma}{2} \int_{\sigma}^{1-\sigma} \tilde{G}_3(s)g(s, u(s), v(s)) \, ds + \lambda \frac{\sigma}{2} \int_{\sigma}^{1-\sigma} \tilde{G}_4(s)f(s, u(s), v(s)) \, ds \\
 &\geq \mu \frac{\sigma}{2} \int_{\sigma}^{1-\sigma} \tilde{G}_3(s)(g_{\infty}^{\sigma} - \varepsilon)(u(s)+v(s)) \, ds + \lambda \frac{\sigma}{2} \int_{\sigma}^{1-\sigma} \tilde{G}_4(s)(f_{\infty}^{\sigma} - \varepsilon)(u(s)+v(s)) \, ds \\
 &\geq \mu \frac{\sigma^2}{4} (g_{\infty}^{\sigma} - \varepsilon) \int_{\sigma}^{1-\sigma} \tilde{G}_3(s) \|(u, v)\|_Y \, ds + \lambda \frac{\sigma^2}{4} (f_{\infty}^{\sigma} - \varepsilon) \int_{\sigma}^{1-\sigma} \tilde{G}_4(s) \|(u, v)\|_Y \, ds \\
 &= \left[\mu \frac{\sigma^2}{4} (g_{\infty}^{\sigma} - \varepsilon) C_{\sigma} + \lambda \frac{\sigma^2}{4} (f_{\infty}^{\sigma} - \varepsilon) D_{\sigma} \right] \|(u, v)\|_Y \\
 &\geq [(1 - a)(1 - \alpha_2) + (1 - a)\alpha_2] \|(u, v)\|_Y = (1 - a) \|(u, v)\|_Y, \quad t \in [0, 1].
 \end{aligned}$$

Therefore,

$$\|T_2(u, v)\| \geq T_2(u, v)(\sigma) \geq a \|(u, v)\|_Y, \quad (u, v) \in P \cap \partial\Omega_2.$$

Further, for $(u, v) \in P \cap \partial\Omega_2$, we have

$$\begin{aligned}
 \|Q(u, v)\|_Y &= \|T_1(u, v)\| + \|T_2(u, v)\| \geq a \|(u, v)\|_Y + (1 - a) \|(u, v)\|_Y \\
 &= \|(u, v)\|_Y.
 \end{aligned}$$

By using Lemma 6 and Theorem 1, we conclude that Q has a fixed point $(u, v) \in P \cap (\Omega_2 \setminus \Omega_1)$ such that $R_1 \leq \|u\| + \|v\| \leq R_2$. That is, (u, v) is a positive solution for problem (1)–(2). \square

Next, we let $L'_2 = \min\{b/(f_0A), (1-b)/(f_0D)\}$ and $L'_4 = \min\{b/(g_0B), (1-b)/(g_0C)\}$. Similar to the proof of Theorem 2, we can easily get the following results.

Theorem 3. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $f_0 = 0$ and $f_\infty^\sigma, g_0, g_\infty^\sigma \in (0, \infty)$, $\alpha_1, \alpha_2 \in [0, 1]$, $a \in [0, 1]$, and $b \in (0, 1)$, then for each $\lambda \in (L_1, \infty)$ and $\mu \in (L_3, L'_4)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

Theorem 4. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $g_0 = 0$ and $f_\infty^\sigma, f_0, g_\infty^\sigma \in (0, \infty)$, $\alpha_1, \alpha_2 \in [0, 1]$, $a \in [0, 1]$, and $b \in (0, 1)$, then for each $\lambda \in (L_1, L'_2)$ and $\mu \in (L_3, \infty)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

Theorem 5. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $f_0 = g_0 = 0$ and $f_\infty^\sigma, g_\infty^\sigma \in (0, \infty)$, $\alpha_1, \alpha_2 \in [0, 1]$, $a \in [0, 1]$, then for each $\lambda \in (L_1, \infty)$ and $\mu \in (L_3, \infty)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

Theorem 6. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $f_0, g_0, f_\infty^\sigma \in (0, \infty)$, $g_\infty^\sigma = \infty$ or $f_0, g_0, g_\infty^\sigma \in (0, \infty)$, $f_\infty^\sigma = \infty$ or $f_0, g_0 \in (0, \infty)$, $f_\infty^\sigma = g_\infty^\sigma = \infty$, $\alpha_3, \alpha_4 \in (0, 1)$, $b \in (0, 1)$, then for each $\lambda \in (0, L_2)$ and $\mu \in (0, L_4)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

Theorem 7. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $f_0 = 0, g_0, f_\infty^\sigma \in (0, \infty)$, $g_\infty^\sigma = \infty$ or $f_0 = 0, g_0, g_\infty^\sigma \in (0, \infty)$, $f_\infty^\sigma = \infty$ or $f_0 = 0, g_0 \in (0, \infty)$, $f_\infty^\sigma = g_\infty^\sigma = \infty$, $b \in (0, 1)$, then for each $\lambda \in (0, \infty)$ and $\mu \in (0, L'_4)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

Theorem 8. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $f_0, f_\infty^\sigma \in (0, \infty)$, $g_0 = 0$, $g_\infty^\sigma = \infty$ or $f_0, g_\infty^\sigma \in (0, \infty)$, $g_0 = 0$, $f_\infty^\sigma = \infty$ or $f_0 \in (0, \infty)$, $g_0 = 0$, $f_\infty^\sigma = g_\infty^\sigma = \infty$, $b \in (0, 1)$, then for each $\lambda \in (0, L'_2)$ and $\mu \in (0, \infty)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

Theorem 9. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $f_0 = g_0 = 0$, $f_\infty^\sigma \in (0, \infty)$, $g_\infty^\sigma = \infty$ or $f_0 = g_0 = 0$, $g_\infty^\sigma \in (0, \infty)$, $f_\infty^\sigma = \infty$ or $f_0 = g_0 = 0$, $f_\infty^\sigma = g_\infty^\sigma = \infty$, then for each $\lambda \in (0, \infty)$ and $\mu \in (0, \infty)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

In order to get the other results, we introduce the extreme limits below

$$f_0^\sigma = \lim_{u+v \rightarrow 0^+} \min_{t \in [\sigma, 1-\sigma]} \frac{f(t, u, v)}{u+v}, \quad g_0^\sigma = \lim_{u+v \rightarrow 0^+} \min_{t \in [\sigma, 1-\sigma]} \frac{g(t, u, v)}{u+v};$$

$$f_\infty^\sigma = \lim_{u+v \rightarrow \infty} \max_{t \in [0, 1]} \frac{f(t, u, v)}{u+v}, \quad g_\infty^\sigma = \lim_{u+v \rightarrow \infty} \max_{t \in [0, 1]} \frac{g(t, u, v)}{u+v}.$$

Theorem 10. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$, $f_0^\sigma, f_\infty, g_0^\sigma, g_\infty \in (0, \infty)$, $\alpha_1, \alpha_2 \in [0, 1]$, $\alpha_3, \alpha_4 \in (0, 1)$, $a \in [0, 1]$, $b \in (0, 1)$, $\tilde{L}_1 < \tilde{L}_2$, and $\tilde{L}_3 < \tilde{L}_4$. Then for each $\lambda \in (\tilde{L}_1, \tilde{L}_2)$ and $\mu \in (\tilde{L}_3, \tilde{L}_4)$, there exists a positive solution $(u(t), v(t))$, $t \in [0, 1]$, for (1)–(2), where

$$\begin{aligned} \tilde{L}_1 &= \max \left\{ \frac{4a\alpha_1}{\sigma^2 f_0^\sigma A_\sigma}, \frac{4(1-a)\alpha_2}{\sigma^2 f_0^\sigma D_\sigma} \right\}, & \tilde{L}_2 &= \min \left\{ \frac{b\alpha_3}{f_\infty A}, \frac{(1-b)\alpha_4}{f_\infty D} \right\}, \\ \tilde{L}_3 &= \max \left\{ \frac{4a(1-\alpha_1)}{\sigma^2 g_0^\sigma B_\sigma}, \frac{4(1-a)(1-\alpha_2)}{\sigma^2 g_0^\sigma C_\sigma} \right\}, & \tilde{L}_4 &= \min \left\{ \frac{b(1-\alpha_3)}{g_\infty B}, \frac{(1-b)(1-\alpha_4)}{g_\infty C} \right\}. \end{aligned}$$

Proof. For σ given in theorem, we consider the above cone $P \subset Y$ and the operators T_1 , T_2 , and Q . Let $\lambda \in (\tilde{L}_1, \tilde{L}_2)$ and $\mu \in (\tilde{L}_3, \tilde{L}_4)$, and let a number $\varepsilon > 0$ be such that $\varepsilon < f_0^\sigma$, $\varepsilon < g_0^\sigma$, and

$$\begin{aligned} \frac{4a\alpha_1}{\sigma^2(f_0^\sigma - \varepsilon)A_\sigma} &\leq \lambda, & \frac{4a(1-\alpha_1)}{\sigma^2(g_0^\sigma - \varepsilon)B_\sigma} &\leq \mu, \\ \frac{4(1-a)\alpha_2}{\sigma^2(f_0^\sigma - \varepsilon)D_\sigma} &\leq \lambda, & \frac{4(1-a)(1-\alpha_2)}{\sigma^2(g_0^\sigma - \varepsilon)C_\sigma} &\leq \mu, \\ \frac{b\alpha_3}{(f_\infty + \varepsilon)A} &\geq \lambda, & \frac{b(1-\alpha_3)}{(g_\infty + \varepsilon)B} &\geq \mu, \\ \frac{(1-b)\alpha_4}{(f_\infty + \varepsilon)D} &\geq \lambda, & \frac{(1-b)(1-\alpha_4)}{(g_\infty + \varepsilon)C} &\geq \mu. \end{aligned}$$

By using (H2) and the definitions of f_0^σ and g_0^σ , we deduce that there exists $R_3 > 0$ such that for all $t \in [\sigma, 1 - \sigma]$, $u, v \in \mathbb{R}^+$ with $0 \leq u + v \leq R_3$, we have $f(t, u, v) \geq (f_0^\sigma - \varepsilon)(u + v)$ and $g(t, u, v) \geq (g_0^\sigma - \varepsilon)(u + v)$. We define the set $\Omega_3 = \{(u, v) \in Y : \|(u, v)\|_Y < R_3\}$. Now let $(u, v) \in P$ with $\|(u, v)\|_Y = R_3$ because $u(t) + v(t) \leq \|u\| + \|v\| = R_3$ for all $t \in [0, 1]$, then by Lemma 4 we obtain

$$\begin{aligned} &T_1(u, v)(\sigma) \\ &\geq \lambda \frac{\sigma}{2} \int_0^1 \tilde{G}_1(s) f(s, u(s), v(s)) \, ds + \mu \frac{\sigma}{2} \int_0^1 \tilde{G}_2(s) g(s, u(s), v(s)) \, ds \\ &\geq \lambda \frac{\sigma}{2} \int_\sigma^{1-\sigma} \tilde{G}_1(s) f(s, u(s), v(s)) \, ds + \mu \frac{\sigma}{2} \int_\sigma^{1-\sigma} \tilde{G}_2(s) g(s, u(s), v(s)) \, ds \\ &\geq \lambda \frac{\sigma}{2} \int_\sigma^{1-\sigma} \tilde{G}_1(s) (f_0^\sigma - \varepsilon) (u(s) + v(s)) \, ds + \mu \frac{\sigma}{2} \int_\sigma^{1-\sigma} \tilde{G}_2(s) (g_0^\sigma - \varepsilon) (u(s) + v(s)) \, ds \\ &\geq \lambda \frac{\sigma^2}{4} (f_0^\sigma - \varepsilon) \int_\sigma^{1-\sigma} \tilde{G}_1(s) \|(u, v)\|_Y \, ds + \mu \frac{\sigma^2}{4} (g_0^\sigma - \varepsilon) \int_\sigma^{1-\sigma} \tilde{G}_2(s) \|(u, v)\|_Y \, ds \end{aligned}$$

$$\begin{aligned}
&= \left[\lambda \frac{\sigma^2}{4} (f_0^\sigma - \varepsilon) A_\sigma + \mu \frac{\sigma^2}{4} (g_0^\sigma - \varepsilon) B_\sigma \right] \|(u, v)\|_Y \\
&\geq [a\alpha_1 + a(1 - \alpha_1)] \|(u, v)\|_Y = a \|(u, v)\|_Y.
\end{aligned}$$

Therefore,

$$\|T_1(u, v)\| \geq T_1(u, v)(\sigma) \geq a \|(u, v)\|_Y, \quad (u, v) \in P \cap \partial\Omega_3.$$

In a similar method, we deduce

$$\begin{aligned}
&T_2(u, v)(\sigma) \\
&\geq \mu \frac{\sigma}{2} \int_0^1 \tilde{G}_3(s) g(s, u(s), v(s)) \, ds + \lambda \frac{\sigma}{2} \int_0^1 \tilde{G}_4(s) f(s, u(s), v(s)) \, ds \\
&\geq \mu \frac{\sigma}{2} \int_\sigma^{1-\sigma} \tilde{G}_3(s) g(s, u(s), v(s)) \, ds + \lambda \frac{\sigma}{2} \int_\sigma^{1-\sigma} \tilde{G}_4(s) f(s, u(s), v(s)) \, ds \\
&\geq \mu \frac{\sigma}{2} \int_\sigma^{1-\sigma} \tilde{G}_3(s) (g_0^\sigma - \varepsilon) (u(s) + v(s)) \, ds + \lambda \frac{\sigma}{2} \int_\sigma^{1-\sigma} \tilde{G}_4(s) (f_0^\sigma - \varepsilon) (u(s) + v(s)) \, ds \\
&\geq \mu \frac{\sigma^2}{4} (g_0^\sigma - \varepsilon) \int_\sigma^{1-\sigma} \tilde{G}_3(s) \|(u, v)\|_Y \, ds + \lambda \frac{\sigma^2}{4} (f_0^\sigma - \varepsilon) \int_\sigma^{1-\sigma} \tilde{G}_4(s) \|(u, v)\|_Y \, ds \\
&= \left[\mu \frac{\sigma^2}{4} (g_0^\sigma - \varepsilon) C_\sigma + \lambda \frac{\sigma^2}{4} (f_0^\sigma - \varepsilon) D_\sigma \right] \|(u, v)\|_Y \\
&\geq [(1 - a)(1 - \alpha_2) + (1 - a)\alpha_2] \|(u, v)\|_Y = (1 - a) \|(u, v)\|_Y, \quad t \in [0, 1].
\end{aligned}$$

Therefore,

$$\|T_2(u, v)\| \geq T_2(u, v)(\sigma) \geq (1 - a) \|(u, v)\|_Y, \quad (u, v) \in P \cap \partial\Omega_3.$$

So, for $(u, v) \in P \cap \partial\Omega_3$, we deduce

$$\begin{aligned}
\|Q(u, v)\|_Y &= \|T_1(u, v)\| + \|T_2(u, v)\| \geq a \|(u, v)\|_Y + (1 - a) \|(u, v)\|_Y \\
&= \|(u, v)\|_Y.
\end{aligned}$$

Now, we define the functions $f_1, g_1 : [0, 1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$, $f_1(t, x) = \max_{0 \leq u+v \leq x} f(t, u, v)$, $g_1(t, x) = \max_{0 \leq u+v \leq x} g(t, u, v)$, $t \in [0, 1]$, $x \in \mathbb{R}^+$. Obviously, $f(t, u, v) \leq f_1(t, x)$, $g(t, u, v) \leq g_1(t, x)$ for all $t \in [0, 1]$, $u \geq 0$, $v \geq 0$, and $u + v \leq x$. Then the functions $f_1(t, \cdot)$ and $g_1(t, \cdot)$ are nondecreasing for every $t \in [0, 1]$, and they satisfy the conditions

$$\limsup_{x \rightarrow \infty} \max_{t \in [0, 1]} \frac{f_1(t, x)}{x} \leq f_\infty, \quad \limsup_{x \rightarrow \infty} \max_{t \in [0, 1]} \frac{g_1(t, x)}{x} \leq g_\infty.$$

Therefore, for $\varepsilon > 0$, there exist $R'_4 > 0$ such that for all $x \geq R'_4$ and $t \in [0, 1]$, we have

$$\begin{aligned} \frac{f_1(t, x)}{x} &\leq \limsup_{x \rightarrow \infty} \max_{t \in [0, 1]} \frac{f_1(t, x)}{x} + \varepsilon \leq f_\infty + \varepsilon, \\ \frac{g_1(t, x)}{x} &\leq \limsup_{x \rightarrow \infty} \max_{t \in [0, 1]} \frac{g_1(t, x)}{x} + \varepsilon \leq g_\infty + \varepsilon, \end{aligned}$$

then $f_1(t, x) \leq (f_\infty + \varepsilon)x$ and $g_1(t, x) \leq (g_\infty + \varepsilon)x$.

We take $R_4 = \max\{2R_3, R'_4\}$, and we denote by $\Omega_4 = \{(u, v) \in Y : \|(u, v)\|_Y < R_4\}$. For $(u, v) \in P \cap \partial\Omega_4$, by the definitions of f_1, g_1 we obtain

$$f(t, u(t), v(t)) \leq f_1(t, \|(u, v)\|_Y), \quad g(t, u(t), v(t)) \leq g_1(t, \|(u, v)\|_Y) \quad \forall t \in [0, 1].$$

Then for all $t \in [0, 1]$, we conclude that

$$\begin{aligned} T_1(u, v)(t) &= \lambda \int_0^1 G_1(t, s) f(s, u(s), v(s)) \, ds + \mu \int_0^1 G_2(t, s) g(s, u(s), v(s)) \, ds \\ &\leq \lambda \int_0^1 \tilde{G}_1(s) f(s, u(s), v(s)) \, ds + \mu \int_0^1 \tilde{G}_2(s) g(s, u(s), v(s)) \, ds \\ &\leq \lambda \int_0^1 \tilde{G}_1(s) f_1(s, \|(u, v)\|_Y) \, ds + \mu \int_0^1 \tilde{G}_2(s) g_1(s, \|(u, v)\|_Y) \, ds \\ &\leq \lambda (f_\infty + \varepsilon) \int_0^1 \tilde{G}_1(s) \|(u, v)\|_Y \, ds + \mu (g_\infty + \varepsilon) \int_0^1 \tilde{G}_2(s) \|(u, v)\|_Y \, ds \\ &\leq [b\alpha_3 + b(1 - \alpha_3)] \|(u, v)\|_Y = b \|(u, v)\|_Y, \quad t \in [0, 1]. \end{aligned}$$

Therefore,

$$\|T_1(u, v)\| \leq b \|(u, v)\|_Y, \quad (u, v) \in P \cap \partial\Omega_4.$$

In a similar method, we obtain

$$\begin{aligned} T_2(u, v)(t) &= \mu \int_0^1 G_3(t, s) g(s, u(s), v(s)) \, ds + \lambda \int_0^1 G_4(t, s) f(s, u(s), v(s)) \, ds \\ &\leq \mu \int_0^1 \tilde{G}_3(s) g(s, u(s), v(s)) \, ds + \lambda \int_0^1 \tilde{G}_4(s) f(s, u(s), v(s)) \, ds \\ &\leq \mu \int_0^1 \tilde{G}_3(s) g_1(s, \|(u, v)\|_Y) \, ds + \lambda \int_0^1 \tilde{G}_4(s) f_1(s, \|(u, v)\|_Y) \, ds \end{aligned}$$

$$\begin{aligned} &\leq \mu(g_\infty + \varepsilon) \int_0^1 \tilde{G}_3(s) \|(u, v)\|_Y ds + \lambda(f_\infty + \varepsilon) \int_0^1 \tilde{G}_4(s) \|(u, v)\|_Y ds \\ &\leq [(1-b)(1-\alpha_4) + (1-b)\alpha_4] \|(u, v)\|_Y = (1-b) \|(u, v)\|_Y, \quad t \in [0, 1]. \end{aligned}$$

Therefore,

$$\|T_2(u, v)\| \leq (1-b) \|(u, v)\|_Y, \quad (u, v) \in P \cap \partial\Omega_4.$$

So, for $(u, v) \in P \cap \partial\Omega_4$, we deduce

$$\begin{aligned} \|Q(u, v)\|_Y &= \|T_1(u, v)\| + \|T_2(u, v)\| \leq b \|(u, v)\|_Y + (1-b) \|(u, v)\|_Y \\ &= \|(u, v)\|_Y. \end{aligned}$$

By using Lemma 6 and Theorem 1, we conclude that Q has a fixed point $(u, v) \in P \cap (\bar{\Omega}_4 \setminus \Omega_3)$ such that $R_3 \leq \|u\| + \|v\| \leq R_4$. That is, (u, v) is a positive solution for problem (1)–(2). \square

Next, let $\tilde{L}'_2 = \min\{b/(f_\infty A), (1-b)/(f_\infty D)\}$ and $\tilde{L}'_4 = \min\{b/(g_\infty B), (1-b)/(g_\infty C)\}$. Similar to the proof of Theorem 10, we can easily get the following results.

Theorem 11. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $f_0^\sigma, g_0^\sigma, f_\infty \in (0, \infty)$ and $g_\infty = 0$, $\alpha_1, \alpha_2 \in [0, 1]$, $a \in [0, 1]$, and $b \in (0, 1)$, then for each $\lambda \in (\tilde{L}_1, \tilde{L}'_2)$ and $\mu \in (\tilde{L}_3, \infty)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

Theorem 12. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $f_0^\sigma, g_0^\sigma, g_\infty \in (0, \infty)$ and $f_\infty = 0$, $\alpha_1, \alpha_2 \in [0, 1]$, $a \in [0, 1]$, and $b \in (0, 1)$, for each $\lambda \in (\tilde{L}_1, \infty)$ and $\mu \in (\tilde{L}_3, \tilde{L}'_4)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

Theorem 13. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $f_0^\sigma, g_0^\sigma \in (0, \infty)$ and $f_\infty = g_\infty = 0$, $\alpha_1, \alpha_2 \in [0, 1]$, $a \in [0, 1]$, for each $\lambda \in (\tilde{L}_1, \infty)$ and $\mu \in (\tilde{L}_3, \infty)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

Theorem 14. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $g_0^\sigma, f_\infty, g_\infty \in (0, \infty)$, $f_0^\sigma = \infty$ or $f_0^\sigma, f_\infty, g_\infty \in (0, \infty)$, $g_0^\sigma = \infty$ or $f_\infty, g_\infty \in (0, \infty)$, $f_0^\sigma = g_0^\sigma = \infty$, $\alpha_3, \alpha_4 \in (0, 1)$, $b \in (0, 1)$, then for each $\lambda \in (0, \tilde{L}_2)$ and $\mu \in (0, \tilde{L}_4)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

Theorem 15. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $g_0^\sigma, f_\infty, g_\infty \in (0, \infty)$, $f_0^\sigma = \infty$ or $f_0^\sigma, f_\infty \in (0, \infty)$, $g_0^\sigma = \infty, g_\infty = 0$ or $f_0^\sigma, g_0^\sigma = \infty, f_\infty \in (0, \infty), g_\infty = 0$, $b \in (0, 1)$, then for each $\lambda \in (0, \tilde{L}'_2)$ and $\mu \in (0, \infty)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

Theorem 16. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $f_\infty = 0, g_0^\sigma, g_\infty \in (0, \infty)$, $f_0^\sigma = \infty$ or $f_0^\sigma, g_\infty \in (0, \infty)$, $g_0^\sigma = \infty, f_\infty = 0$ or $f_\infty = 0, g_\infty \in (0, \infty)$, $f_0^\sigma = g_0^\sigma = \infty, b \in (0, 1)$, then for each $\lambda \in (0, \infty)$ and $\mu \in (0, \tilde{L}'_4)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

Theorem 17. Assume that (H1) and (H2) hold, $\sigma \in (0, 1/2)$. If $f_\infty = g_\infty = 0, g_0^\sigma \in (0, \infty)$, $f_\infty^\sigma = \infty$ or $f_0^\sigma \in (0, \infty)$, $g_0^\sigma = \infty, f_\infty = g_\infty = 0$ or $f_0^\sigma = g_0^\sigma = \infty, f_\infty = g_\infty = 0$, then for each $\lambda \in (0, \infty)$ and $\mu \in (0, \infty)$, there exists a positive solution $(u(t), v(t))$ for problem (1)–(2).

4 Examples

We consider the system of differential equations

$$\begin{aligned} u^{(4)}(t) &= \lambda f(t, u(t), v(t)), & t \in [0, 1], \\ v^{(4)}(t) &= \mu g(t, u(t), v(t)), & t \in [0, 1], \end{aligned} \tag{11}$$

subject to the coupled integral boundary conditions

$$\begin{aligned} u(0) = u'(1) = u'''(1) = 0, & \quad u''(0) = \int_0^1 s v''(s) \, ds, \\ v(0) = v'(1) = v'''(1) = 0, & \quad v''(0) = \int_0^1 \left(\frac{1}{2} s^2 - \frac{1}{3} s^{3/2} \right) u''(s) \, ds. \end{aligned} \tag{12}$$

Then we have

$$h_1(t) = t, \quad h_2(t) = \frac{1}{2}t^2 - \frac{1}{3}t^{3/2}, \quad \Delta = \int_0^1 s \, ds \int_0^1 \left(\frac{1}{2} s^2 - \frac{1}{3} s^{3/2} \right) \, ds = \frac{1}{60},$$

$$G_1(t, s) = g_1(t, s) + \frac{30}{59}(2t - t^2) \int_0^1 g_2(\tau, s) \left(\frac{1}{2} \tau^2 - \frac{1}{3} \tau^{3/2} \right) \, d\tau, \quad t, s \in [0, 1],$$

$$G_2(t, s) = \frac{60}{59}(2t - t^2) \int_0^1 g_2(\tau, s) \tau \, d\tau, \quad t, s \in [0, 1],$$

$$G_3(t, s) = g_1(t, s) + \frac{2}{59}(2t - t^2) \int_0^1 g_2(\tau, s) \tau \, d\tau, \quad t, s \in [0, 1],$$

$$G_4(t, s) = \frac{60}{59}(2t - t^2) \int_0^1 g_2(\tau, s) \left(\frac{1}{2} \tau^2 - \frac{1}{3} \tau^{3/2} \right) \, d\tau, \quad t, s \in [0, 1],$$

$$\tilde{G}_1(s) = \frac{61}{118}s - \frac{1}{6}s^3 - \frac{5}{236}s^4 - \frac{8}{413}s^{7/2}, \quad \tilde{G}_2(s) = -\frac{30}{59}s^3 + \frac{60}{59}s^2,$$

$$\tilde{G}_3(s) = \frac{1}{2}s - \frac{23}{118}s^3 + \frac{2}{59}s^2, \quad \tilde{G}_4(s) = -\frac{5}{118}s^4 + \frac{16}{433}s^{7/2} + \frac{2}{59}s,$$

where $g_i(t, s)$ ($i = 1, 2$) are defined in Lemma 1.

Take $\sigma = 1/3$, after some computations, we obtain that

$$A = \int_0^1 \tilde{G}_1(s) \, ds \approx 0.4336, \quad B = \int_0^1 \tilde{G}_2(s) \, ds \approx 0.8475,$$

$$\begin{aligned}
C &= \int_0^1 \tilde{G}_3(s) \, ds \approx 0.0458, & D &= \int_0^1 \tilde{G}_4(s) \, ds \approx 0.0001, \\
A_\sigma &= \int_{1/3}^{2/3} \tilde{G}_1(s) \, ds \approx 0.0472, & B_\sigma &= \int_{1/3}^{2/3} \tilde{G}_2(s) \, ds \approx 0.0675, \\
C_\sigma &= \int_{1/3}^{2/3} \tilde{G}_3(s) \, ds \approx 0.047, & D_\sigma &= \int_{1/3}^{2/3} \tilde{G}_4(s) \, ds \approx 0.0049.
\end{aligned}$$

Example 1. In (11), we consider the functions

$$\begin{aligned}
f(t, u, v) &= \frac{1}{(1+t)^2} \left[\frac{(u+v)}{5[1+\ln(1+u+v)]} + \frac{5(u+v)^2}{2(1+u+v)} \right], \\
g(t, u, v) &= \sqrt[3]{2-t} \left[\frac{1}{11} \sin(u+v) + 1000(u+v) \arctan(u+v) \right],
\end{aligned}$$

then we have $f_0 = 1/5$, $g_0 = 1/11$, $f_\infty = 9/10$, $g_\infty = 500\pi$. Take $a = b = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/2$, then we conclude that $L_1 \approx 0.0483$, $L_2 \approx 2.883$, $L_3 \approx 0.122$, $L_4 \approx 3.2448$. Therefore, by Theorem 2, for each $\lambda \in (0.0483, 2.883)$, $\mu \in (0.122, 3.2448)$, there exists a positive solution $(u(t), v(t))$, $t \in [0, 1]$, for problem (11)–(12).

Example 2. In (11), we consider the functions

$$\begin{aligned}
f(t, u, v) &= \sqrt[3]{1-t} \left[\frac{(u+v)[2000 \cos(u+v) + 1/4(u+v+2)]}{u+v+1} \right], \\
g(t, u, v) &= \frac{1}{1+t} \left[\frac{5}{22}(u+v) + 500 \ln(1+u+v) \right],
\end{aligned}$$

then we have $f_0^\sigma = 2667/2\sqrt{3}$, $g_0^\sigma = 6605/22$, $f_\infty = \sqrt{2}/50$, $g_\infty = 5/22$. Take $a = b = \alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/2$, then we conclude that $\tilde{L}_1 \approx 0.7952$, $\tilde{L}_2 \approx 1.6308$, $\tilde{L}_3 \approx 0.638$, $\tilde{L}_4 \approx 1.298$. Therefore, by Theorem 10, for each $\lambda \in (0.7952, 1.6308)$, $\mu \in (0.638, 1.298)$, there exists a positive solution $(u(t), v(t))$, $t \in [0, 1]$, for problem (11)–(12).

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