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Bogdanov–Takens and triple zero bifurcations in general differential systems with m delays*

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Abstract. This paper mainly concerns the derivation of the normal forms of the Bogdanov–Takens (BT) and triple zero bifurcations for differential systems with m discrete delays. The feasible algorithms to determine the existence of the corresponding bifurcations of the system at the origin are given. By using center manifold reduction and normal form theory, the coefficient formulas of normal forms are derived and some examples are presented to illustrate our main results.

Keywords: differential system, delays, Bogdanov–Takens bifurcation, triple zero bifurcation, normal form.

1 Introduction

Many researchers have studied some kinds of codimension bifurcation phenomena for some delayed differential systems, these bifurcation phonmena include saddle-node bifurcation, Hopf bifurcation, Hopf-zero bifurcation, double Hopf bifurcation and so on. And there are some results about the Bogdanov–Takens (BT) bifurcation (a double zero eigenvalue with geometric multiplicity one) and triple zero (a triple zero eigenvalue with geometric multiplicity one) bifurcation for general differential systems with one delay, one can see, for example, [4, 6, 16, 20, 21], and their results can be used to study the codimension bifurcation of some predator–prey systems, neural networks models, Van der Pol's oscillator, etc. One can see, for example, [3, 9–12, 14, 15, 18, 19, 23].

Notice that the authors in [16, 20] have given some feasible formulas to determine the BT singularity, triple zero singularity, and the generalized eigenspace associated with

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zero eigenvalue in $\mathbb{R}^n$ for the following system:

$$\dot{x} = A(\mathbf{P})x(t) + B(\mathbf{P})x(t-1) + F\big(x(t), x(t-1), \mathbf{P}\big),$$

where $\mathbf{P} \in \mathbb{R}^q$ ($q = 2$ or 3) is a parameter vector, $x \in \mathbb{R}^n$. By using center manifold reduction and normal form theory, the concrete normal forms (two or three dimension ordinary differential equations) of the parameterized delay differential systems with BT and triple zero bifurcations at the origin were obtained.

Now the general coefficient formulas of normal forms corresponding to BT and triple zero bifurcations for general differential systems with many discrete delays have not been given except for some special differential systems as discussed in [1, 13, 17, 22]. In this paper, we will generalize and apply these methods used in [16, 20] to deduce the normal forms of BT and triple zero bifurcations of the following system with m delays:

$$\dot{x} = A(\mathbf{P})x(t) + \sum_{l=1}^{m} B_l(\mathbf{P})x(t-\tau_l) + F\big(x(t), x(t-\tau_1), \dots, x(t-\tau_m), \mathbf{P}\big). \tag{1}$$

The organization of this paper is as follows. In Section 2, the existence conditions of BT and triple zero singularities for general systems with m delays will be given. In Sections 3 and 4, by applying the center manifold theorem and normal form theory, the corresponding coefficient formulas of the normal forms for delay differential system are obtained. In Section 5, a real application is exhibited.

2 Stability of the trivial equilibrium

To study the BT and triple zero bifurcations of system (1), we first give the similar assumptions as the authors used in [16, 20]:

(H1) $A(\mathbf{P})$, $B_l(\mathbf{P})$ ($l = 1, 2, \dots, m$) are C^r ($r \geqslant 2$) smooth matrix-valued functions from $\mathbb{R}^q$ to $\mathbb{R}^{n\times n}$, $F(x, y_1, y_2, \dots, y_m, \mathbf{P})$ is a C^r ($r \geqslant 2$) smooth function from $\underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{m+1} \times \mathbb{R}^q$ to $\mathbb{R}^n$ and, for all $\mathbf{P} \in \mathbb{R}^q$, satisfies

$$F(\underbrace{0, 0, 0, \dots, 0}_{m+1}, \mathbf{P}) = 0, \qquad \frac{\partial F}{\partial x}(\underbrace{0, 0, 0, \dots, 0}_{m+1}, \mathbf{P}) = 0,$$
$$\frac{\partial F}{\partial y_l}(\underbrace{0, 0, 0, \dots, 0}_{m+1}, \mathbf{P}) = 0 \quad (l = 1, 2, \dots, m).$$

Without loss of generality, we assume $-\tau_1 < -\tau_2 < \dots < -\tau_m < 0$, then we denote the Banach space of continuous mapping from $[-\tau_1, 0]$ to $\mathbb{R}^n$ with norm $\|\phi\| = \max_{\theta\in[-\tau_1,0]} |\phi(\theta)|$ by $C = C([-\tau_1, 0], \mathbb{R}^n)$. Let

$$L(\mathbf{P})x_t = \int_{-\tau_1}^{0} \big[\mathrm{d}\eta_{\mathbf{P}}(\theta)\big]x_t(\theta) \triangleq A(\mathbf{P})x(t) + \sum_{l=1}^{m} B_l(\mathbf{P})x(t-\tau_l), \tag{2}$$

in which $x_t(\theta) = x(t+\theta)$, and $\eta_{\mathbf{P}}(\theta)$ is a bounded variation matrix-valued function on $[-\tau_1, 0]$. Especially, when $\mathbf{P} = 0$, we have $A = A(0)$, $B_1 = B_1(0)$, $B_2 = B_2(0)$, $\dots$, $B_m = B_m(0)$, and $L(0)x_t = Ax(t) + \sum_{l=1}^m B_l x(t-\tau_l) \triangleq L_0 x(t)$. From the definition of L_0 we can obtain that $L_0(\xi) = (A + \sum_{l=1}^m B_l)\xi$, $L_0(\theta^j \xi) = (-1)^j \sum_{l=1}^m \tau_l^j B_l \xi$, $L_0(\mathrm{e}^{\lambda\theta}\xi) = (A + \sum_{l=1}^m B_l \mathrm{e}^{-\lambda\tau_l})\xi$, where $j \in N^+$, for all $\xi \in \mathbb{R}^n$. These formulas will be used frequently in the rest of this paper.

We rewrite system (1) as the following functional differential equation (FDE):

$$\dot{x}(t) = L(\mathbf{P})x_t + F(x_t, \mathbf{P}), \tag{3}$$

which can be linearized at $(x_t, \mathbf{P}) = (0, 0)$ as

$$\dot{x}(t) = L_0 x_t. \tag{4}$$

From [7, 8] a C_0-semigroup $\{T_0(t),\, t \geqslant 0\}$ on C can be defined by the fundamental solution of system (4) with infinitesimal generator $\mathcal{A}_0 : C \to C$:

$$\mathcal{A}_0\phi = \dot{\phi}, \quad D(\mathcal{A}_0) = \left\{\phi \in C^1\big([-\tau_1, 0], \mathbb{R}^n\big)\colon\ \dot{\phi}(0) = \int\limits_{-\tau_1}^{0} \big[\mathrm{d}\eta_0(\theta)\big]\phi(\theta) = L_0\phi\right\}.$$

With the definition of $\mathcal{A}_0$, system (4) is equivalent to an abstract ordinary differential equation (ODE) $\dot{x} = \mathcal{A}_0 x$ in C. Furthermore, the spectrum of the operator $\mathcal{A}_0$ consists of its point spectrum, i.e. $\sigma(\mathcal{A}_0) = \sigma_p(\mathcal{A}_0) = \{\lambda\colon\ \Delta_0(\lambda) = 0\}$, the characteristic equation of system (4) is

$$\Delta_0(\lambda) \triangleq \det\big(\lambda I - L_0\big(\mathrm{e}^{\lambda\theta} I\big)\big) = \det\left(\lambda I - A - \sum_{l=1}^{m} B_l \mathrm{e}^{-\tau_l\lambda}\right) = 0.$$

To study system (1), we further make the following assumptions:

(H2) $\operatorname{Re}\lambda \neq 0$ if $\lambda \in \sigma_p(\mathcal{A}_0) \setminus \{0\}$;
(H3) $\lambda = 0$ is the eigenvalue of $\mathcal{A}_0$ with algebraic multiplicity 2 (3) and geometric multiplicity 1.

One can see that system (1) has a BT (triple zero) singularity if (H1)–(H3) hold. Then we will give an equivalent description for BT and triple zero singularities in system (1).

Theorem 1. *Let* (H1), (H2) *hold, the delay differential system* (1) *has a BT singularity when it satisfies the following conditions*:

(i) $\operatorname{rank}(A + \sum_{l=1}^m B_l) = n - 1$,
(ii) *if* $N(A + \sum_{l=1}^m B_l) = \operatorname{span}\{\phi_1^0\}$, *then* $\sum_{l=1}^m \tau_l B_l + I\phi_1^0 \in R(A + \sum_{l=1}^m B_l)$,
(iii) *if* $(A + \sum_{l=1}^m B_l)\phi_2^0 = (\sum_{l=1}^m \tau_l B_l + I)\phi_1^0$, *then* $(\sum_{l=1}^m \tau_l B_l + I)\phi_2^0 - \sum_{l=1}^m \tau_l^2 B_l \times \phi_1^0/2 \notin R(A + \sum_{l=1}^m B_l)$, *where* $\phi_1^0, \phi_2^0 \in \mathbb{R}^n$.

Theorem 2. *Let* (H1), (H2) *and* (i), (ii) *in the above theorem hold, the delay differential system* (1) *has a triple zero singularity when it satisfies the following conditions*:

(iv) $(\sum_{l=1}^{m}\tau_l B_l+I)\phi_2^0-\sum_{l=1}^{m}\tau_l^2 B_l\phi_1^0/2\in R(A+\sum_{l=1}^{m}B_l)$ *if* $(A+\sum_{l=1}^{m}B_l)\phi_2^0=(\sum_{l=1}^{m}\tau_l B_l+I)\phi_1^0$,

(v) $(\sum_{l=1}^{m}\tau_l B_l+I)\phi_3^0-\sum_{l=1}^{m}\tau_l^2 B_l\phi_2^0/2+\sum_{l=1}^{m}\tau_l^3 B_l\phi_1^0/6\notin R(A+\sum_{l=1}^{m}B_l)$ *if* $(A+\sum_{l=1}^{m}B_l)\phi_3^0=(\sum_{l=1}^{m}\tau_l B_l+I)\phi_2^0-\sum_{l=1}^{m}\tau_l^2 B_l\phi_1^0/2$, *where* $\phi_1^0,\phi_2^0,\phi_3^0\in\mathbb{R}^n$.

By using the methods used in [4,16,20], we reduce system (1) to an ordinary differential system with dimension q on its center manifold. Rewrite the parameterized system (1) as the following FDE:

$$\dot{x}(t)=L_0x_t+\big[L(\mathbf{P})-L_0\big]x_t+F(x_t,\mathbf{P}),\quad \dot{\mathbf{P}}(t)=0. \tag{5}$$

Define $\widetilde{C}=C([-\tau_1,0],\mathbb{R}^n\times\mathbb{R}^q)$ as its phase space. In addition, let $\tilde{x}(t)=(x(t),\mathbf{P}(t))$ be the solution of (5), then (5) becomes

$$\dot{\tilde{x}}(t)=\widetilde{L}_0\tilde{x}_t+\widetilde{F}(\tilde{x}_t), \tag{6}$$

where the operator $\widetilde{L}_0\tilde{x}_t=(L_0x_t,0)$ is bounded linear and is from $\widetilde{C}$ to $\mathbb{R}^n\times\mathbb{R}^q$, and $\widetilde{F}(\tilde{x}_t)=([L(\mathbf{P}(0))-L_0]x_t+F(x_t,\mathbf{P}(0)),0)\triangleq(\widehat{F}(x_t,\mathbf{P}),0)$ with $x_t\in C,\mathbf{P}\in C_2\triangleq C([-\tau_1,0],\mathbb{R}^q)$. The linearization of (6) at $\tilde{x}_t=0$ is

$$\dot{\tilde{x}}(t)=\widetilde{L}_0\tilde{x}_t. \tag{7}$$

Define the infinitesimal generator of system (7) as $\tilde{\mathcal{A}}_0$, then we have $\tilde{\mathcal{A}}_0=(\mathcal{A}_0,0)$. The eigenvalues of $\tilde{\mathcal{A}}_0$ include all eigenvalues of $\mathcal{A}_0$ and double or triple zero eigenvalues introduced by $\dot{\mathbf{P}}=0$. Let $\tilde{\Lambda}$ be the set of $2q$ zero eigenvalues for system (7).

Similar to [20], we decompose the phase space C of system (3). Let $C=P\oplus Q$, where P is the invariant space of $\mathcal{A}_0$ corresponding to the zero eigenvalue and Q is the complementary space. $C^*=C([0,\tau_1],\mathbb{R}^{n*})$ is defined as the adjoint space of C, where $\mathbb{R}^{n*}$ is the n-dimensional space of row vectors. The bilinear inner product on $C^*\times C$ is defined by

$$(\psi,\phi)=\psi(0)\phi(0)-\int_{-\tau_1}^{0}\int_{0}^{\theta}\psi(\xi-\theta)\,\mathrm{d}_{\eta_0(\theta)}\phi(\xi)\,\mathrm{d}\xi. \tag{8}$$

Under assumption (H3), one can see that the dimension of the space P is q. Let $\Phi(\theta)$ and $\Psi(s)$ be the bases of P and its dual space P^*, respectively. Then $(\Psi,\Phi)=I_q$, where $(\Psi,\Phi)\triangleq(\psi_j,\phi_i)$.

3 Normal form of BT bifurcation

To study the BT bifurcation at the origin of system (1), we need use the following lemma.

Lemma 1. *The bases of* P *and its dual space* P^* *are* $\Phi(\theta)=(\phi_1(\theta),\phi_2(\theta))$, $\Psi(s)=\mathrm{col}(\psi_1(s),\psi_2(s))$, *where* $\phi_1(\theta)=\phi_1^0\in\mathbb{R}^n\setminus\{0\}$, $\phi_2(\theta)=\phi_2^0+\phi_1^0\theta$, $\phi_2^0\in\mathbb{R}^n$ *and*

$\psi_2(s) = \psi_2^0 \in \mathbb{R}^{n*} \setminus \{0\}$, $\psi_1(s) = \psi_1^0 - s\psi_2^0$, $\psi_1^0 \in \mathbb{R}^{n*}$, *which satisfy*

$$\text{(a)}\ \left(A + \sum_{l=1}^{m} B_l\right)\phi_1^0 = 0, \qquad \text{(b)}\ \left(A + \sum_{l=1}^{m} B_l\right)\phi_2^0 = \left(\sum_{l=1}^{m} \tau_l B_l + I\right)\phi_1^0,$$

$$\text{(c)}\ \psi_2^0\left(A + \sum_{l=1}^{m} B_l\right) = 0, \qquad \text{(d)}\ \psi_1^0\left(A + \sum_{l=1}^{m} B_l\right) = \psi_2^0\left(\sum_{l=1}^{m} \tau_l B_l + I\right),$$

$$\text{(e)}\ \psi_1^0\phi_2^0 + \psi_1^0\sum_{l=1}^{m}\tau_l B_l\phi_2^0 - \frac{1}{2}\psi_1^0\sum_{l=1}^{m}\tau_l^2 B_l\phi_1^0 - \frac{1}{2}\psi_2^0\sum_{l=1}^{m}\tau_l^2 B_l\phi_2^0 + \frac{1}{6}\psi_2^0\sum_{l=1}^{m}\tau_l^3 B_l\phi_1^0 = 0,$$

$$\text{(f)}\ \psi_2^0\phi_2^0 + \psi_2^0\sum_{l=1}^{m}\tau_l B_l\phi_2^0 - \frac{1}{2}\psi_2^0\sum_{l=1}^{m}\tau_l^2 B_l\phi_1^0 = 1.$$

Without considering the factors of coefficient constant, we can determine the unique vectors ϕ_1^0, ϕ_2^0, ψ_1^0, ψ_2^0 by (a)–(f).

Proof. By assumption (H3), we know that there are linearly independent functions ϕ_1, $\phi_2 \in C$ satisfying $\mathcal{A}_0\phi_1 = 0$, $\mathcal{A}_0\phi_2 = \phi_1$.

By the definition of operator $\mathcal{A}_0$, $\mathcal{A}_0\phi_1 = 0$ is equivalent to

$$L_0\phi_1(\theta) = 0, \quad \theta = 0,$$
$$\dot{\phi}_1(\theta) = 0, \quad -1 \leqslant \theta \leqslant 0.$$

Hence, we obtain $\phi_1(\theta) = \phi_1^0 \in \mathbb{R}^n \setminus \{0\}$, $(A + \sum_{l=1}^m B_l)\phi_1^0 = 0$. Similarly, $\mathcal{A}_0\phi_2 = \phi_1$ is equivalent to

$$L_0\phi_2(\theta) = \phi_1^0, \quad \theta = 0,$$
$$\dot{\phi}_2(\theta) = \phi_1^0, \quad -1 \leqslant \theta \leqslant 0, \tag{9}$$

which is solved by $\phi_2(\theta) = \phi_2^0 + \phi_1^0\theta$, $\phi_2^0 \in \mathbb{R}^n$, $(A + \sum_{l=1}^m B_l)\phi_2^0 = (\sum_{l=1}^m \tau_l B_l + I)\phi_1^0$.

To prove Lemma 1, it needs to use the adjoint operator $\mathcal{A}_0^* : C^* \to C^*$ of $\mathcal{A}_0$ by

$$\mathcal{A}_0^*\psi = -\dot{\psi}, \quad D(\mathcal{A}_0^*) = \left\{\psi \in C^1\big([-\tau_1, 0], \mathbb{R}^{n*}\big)\colon -\dot{\psi}(0) = \int_{-\tau_1}^{0} \psi(-\theta)\,\mathrm{d}\eta_0(\theta)\right\}.$$

From $\mathcal{A}_0^*\psi_2 = 0$, by

$$-\frac{\mathrm{d}\psi_2}{\mathrm{d}s}(s) = 0, \quad 0 \leqslant s \leqslant 1,$$
$$\int_{-\tau_1}^{0} \psi_2(-\theta)\,\mathrm{d}\eta_0(\theta) = 0, \quad s = 0, \tag{10}$$

we can deduce $\psi_2(s) = \psi_2^0 \in \mathbb{R}^{n*} \setminus \{0\}$, $\psi_2^0(A + \sum_{l=1}^m B_l) = 0$.

Furthermore, by $\mathcal{A}_0^*\psi_1 = \psi_2$ and

$$-\frac{\mathrm{d}\psi_1}{\mathrm{d}s}(s) = \psi_2^0, \quad 0 \leqslant s \leqslant 1, \qquad \int_{-\tau_1}^{0} \psi_1(-\theta)\,\mathrm{d}\eta_0(\theta) = \psi_2^0, \quad s = 0, \tag{11}$$

we deduce $\psi_1(s) = \psi_1^0 - s\psi_2^0$, $\psi_1^0 \in \mathbb{R}^{n*}$, $\psi_1^0(A + \sum_{l=1}^m B_l) = \psi_2^0(\sum_{l=1}^m \tau_l B_l + I)$.

Therefore, by solving (9)–(11), we can easily get (a)–(d) of the lemma. Finally, from $(\Psi, \Phi) = I_2$ and using (8) and (2) we have

$$\begin{aligned}
(\psi_1, \phi_1) &= \psi_1^0\phi_1^0 - \frac{1}{2}\psi_2^0\left(\sum_{l=1}^m \tau_l^2 B_l\right)\phi_1^0 + \psi_1^0\left(\sum_{l=1}^m \tau_l B_l\right)\phi_1^0 = 1,\\
(\psi_2, \phi_2) &= \psi_2^0\phi_2^0 - \frac{1}{2}\psi_2^0\left(\sum_{l=1}^m \tau_l^2 B_l\right)\phi_1^0 + \psi_2^0\left(\sum_{l=1}^m \tau_l B_l\right)\phi_2^0 = 1,\\
(\psi_1, \phi_2) &= \frac{1}{6}\psi_2^0\sum_{l=1}^m \tau_l^3 B_l\phi_1^0 + \psi_1^0\sum_{l=1}^m \tau_l B_l\phi_2^0 - \frac{1}{2}\psi_1^0\sum_{l=1}^m \tau_l^2 B_l\phi_1^0\\
&\quad - \frac{1}{2}\psi_2^0\sum_{l=1}^m \tau_l^2 B_l\phi_2^0 + \psi_1^0\phi_2^0 = 0,\\
(\psi_2, \phi_1) &= \psi_2^0\phi_1^0 + \psi_2^0\left(\sum_{l=1}^m \tau_l B_l\right)\phi_1^0 = 0.
\end{aligned} \tag{12}$$

In fact, the first and the fourth formulae in (12) hold naturally by (a)–(e) of the lemma. The proof is complete. □

It is easy to see that $\Phi(\theta)$ satisfies $\dot{\Phi} = \Phi J$, where

$$J = \begin{pmatrix} 0 & 1\\ 0 & 0 \end{pmatrix}.$$

Denote the Taylor expansion of $\widehat{F}(x_t, \mathbf{P})$ with respect to x_t and $\mathbf{P}$ in system (7) as $\widehat{F}(x_t, \mathbf{P}) = \sum_{w \geqslant 2}(1/w!)\widehat{F}_w(x_t, \mathbf{P})$, we have

$$\begin{aligned}
\frac{1}{2}\widehat{F}_2(x_t, \mathbf{P}) &= A_1 x(t)\alpha_1 + A_2 x(t)\alpha_2 + \sum_{l=1}^m [B_{l1}x(t-\tau_l)\alpha_1 + B_{l2}x(t-\tau_l)\alpha_2]\\
&\quad + \sum_{i=1}^m \sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}x_i(t-\tau_k)x(t-\tau_j),
\end{aligned} \tag{13}$$

where $\tau_0 = 0$ and A_1, A_2, B_{l1}, B_{l2}, D_{ikj} are coefficient matrices. According to the discussion in the above, we also know there is no terms of $O(\mathbf{P}^2)$ in $\widehat{F}_2(x_t, \mathbf{P})$ because $\widehat{F}(0, \mathbf{P}) = 0$ for all $\mathbf{P} \in \mathbb{R}^2$.

From (13) we can get the following system:

$$\begin{aligned}
&\frac{1}{2}\widehat{F}_2(\Phi z,\mathbf{P})\\
&\quad= A_1\big[\Phi(0)(z_1,z_2)^{\mathrm{T}}\big]\alpha_1 + A_2\big[\Phi(0)(z_1,z_2)^{\mathrm{T}}\big]\alpha_2\\
&\qquad+\sum_{l=1}^{m}\big\{B_{l1}\big[\Phi(-\tau_l)(z_1,z_2)^{\mathrm{T}}\big]\alpha_1 + B_{l2}\big[\Phi(-\tau_l)(z_1,z_2)^{\mathrm{T}}\big]\alpha_2\big\}\\
&\qquad+\sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\big[\big(\phi_{1i}(-\tau_k),\phi_{2i}(-\tau_k)\big)(z_1,z_2)^{\mathrm{T}}\big]\big[\Phi(-\tau_j)(z_1,z_2)^{\mathrm{T}}\big],
\end{aligned}$$

where ϕ_{ji} stands for the ith element of ϕ_j.

After some calculations we obtain

$$\begin{aligned}
&\frac{1}{2}\widehat{F}_2(\Phi z,\mathbf{P})\\
&\quad=\left[A_1\phi_1(0)+\sum_{l=1}^{m}B_{l1}\phi_1(-\tau_l)\right]\alpha_1 z_1 + \left[A_2\phi_1(0)+\sum_{l=1}^{m}B_{l2}\phi_1(-\tau_l)\right]\alpha_2 z_1\\
&\qquad+\left[A_1\phi_2(0)+\sum_{l=1}^{m}B_{l1}\phi_2(-\tau_l)\right]\alpha_1 z_2 + \left[A_2\phi_2(0)+\sum_{l=1}^{m}B_{l2}\phi_2(-\tau_l)\right]\alpha_2 z_2\\
&\qquad+\sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\phi_1(-\tau_j)\phi_{1i}(-\tau_k)z_1^2\\
&\qquad+\sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\big[\phi_1(-\tau_j)\phi_{2i}(-\tau_k)+\phi_2(-\tau_j)\phi_{1i}(-\tau_k)\big]z_1 z_2\\
&\qquad+\sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\phi_2(-\tau_j)\phi_{2i}(-\tau_k)z_2^2.
\end{aligned}$$

Since $\phi_1(0)=\phi_1(-\tau_l)=\phi_1^0$, $\phi_2(0)=\phi_2^0$, $\phi_2(-\tau_l)=\phi_2^0-\tau_l\phi_1^0$, $\psi_1(0)=\psi_1^0$, $\psi_2(0)=\psi_2^0$, the above expression can be simplified by substituting them into it. Furthermore, by [20] we also know

$$g_2^1=\big(I-P_{I,2}^1\big)f_2^1,\qquad f_2^1(z,0,\mathbf{P})=\Psi(0)\widehat{F}_2(\Phi z,\mathbf{P}),$$

$$P_{I,2}^1\begin{pmatrix}z_1^2\\0\end{pmatrix}=\begin{pmatrix}z_1^2\\-2z_1z_2\end{pmatrix},\qquad P_{I,2}^1\begin{pmatrix}\alpha_1 z_1\\0\end{pmatrix}=\begin{pmatrix}\alpha_1 z_1\\-\alpha_1 z_2\end{pmatrix},\qquad P_{I,2}^1\begin{pmatrix}\alpha_2 z_1\\0\end{pmatrix}=\begin{pmatrix}\alpha_2 z_1\\-\alpha_2 z_2\end{pmatrix},$$

and for other p, it has

$$\big(I-P_{I,2}^1\big)p=\begin{cases}p, & p\in\operatorname{Im}(M_2^1)^c,\\ 0, & p\in\operatorname{Im}(M_2^1),\end{cases}$$

the bases of $\operatorname{Im}(M_2^1)^c$ and $\operatorname{Im}(M_2^1)$ can be seen in [20].

The normal form for (6) on the center manifold corresponding to the space P can be written as $\dot{z} = Jz + g_2^1(z, 0, \mathbf{P})/2 + \mathrm{h.o.t.}$, (see [20] for detail).

Hence, we have the following theorem.

Theorem 3. *Let* (H1)–(H3) *hold. Then the delay differential system* (1) *can be reduced to the following two-dimensional system of ODE on the center manifold at* $(x_t, \mathbf{P}) = (0, 0)$:

$$\dot{z}_1 = z_2, \qquad \dot{z}_2 = \delta_1 z_1 + \delta_2 z_2 + d_1 z_1^2 + d_2 z_1 z_2 + \mathrm{h.o.t.}, \tag{14}$$

where

$$\delta_1 = \psi_2^0\left(A_1 + \sum_{l=1}^{m} B_{l1}\right)\phi_1^0\alpha_1 + \psi_2^0\left(A_2 + \sum_{l=1}^{m} B_{l2}\right)\phi_1^0\alpha_2,$$

$$\delta_2 = \left\{\psi_1^0\left(A_1 + \sum_{l=1}^{m} B_{l1}\right)\phi_1^0 + \psi_2^0\left[\left(A_1 + \sum_{l=1}^{m} B_{l1}\right)\phi_2^0 - \sum_{l=1}^{m}\tau_l B_{l1}\phi_1^0\right]\right\}\alpha_1 + \left\{\psi_1^0\left(A_2 + \sum_{l=1}^{m} B_{l2}\right)\phi_1^0 + \psi_2^0\left[\left(A_2 + \sum_{l=1}^{m} B_{l2}\right)\phi_2^0 - \sum_{l=1}^{m}\tau_l B_{l2}\phi_1^0\right]\right\}\alpha_2,$$

$$d_1 = \psi_2^0\sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\phi_1^0\phi_{1i}^0,$$

$$d_2 = \psi_2^0\left\{\sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\left(\phi_1^0\phi_{2i}^0 + \phi_2^0\phi_{1i}^0\right) - \sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m}(\tau_k + \tau_j)D_{ikj}\phi_1^0\phi_{1i}^0\right\} + 2\psi_1^0\sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\phi_1^0\phi_{1i}^0.$$

Let

$$\Pi = \begin{pmatrix} \frac{\partial\delta_1}{\partial\alpha_1} & \frac{\partial\delta_1}{\partial\alpha_2} \\ \frac{\partial\delta_2}{\partial\alpha_1} & \frac{\partial\delta_2}{\partial\alpha_2} \end{pmatrix},$$

in addition, we need assume

(H4) $\det\Pi \neq 0$.

In this case, system (14) has two equilibria $E_1 = (0, 0)$ and $E_2 = (-\delta_1/d_1, 0)$. The bifurcation curves near the origin in the α_1 and α_2 parameter space are the following [1, 13]:

- TB: $\delta_1 = 0$ (transcritical bifurcation occurs),
- H_0: $\delta_2 = 0$, $\delta_1 < 0$ (Hopf bifurcation from the zero equilibrium point),
- H_1: $\delta_2 = (d_2/d_1)\delta_1$, $\delta_1 > 0$ (a Hopf bifurcation from equilibrium $-(\delta_1/d_1, 0)$,
- H_c^0: $\delta_2 = (d_2/7d_1)\delta_1$, $\delta_1 < 0$ (a homoclinic bifurcation with the zero equilibrium point),
- H_c^1: $\delta_2 = (6d_2/7d_1)\delta_1$, $\delta_1 > 0$ (a homoclinic bifurcation with the equilibrium $(-\delta_1/d_1, 0)$).

4 Normal form of triple zero bifurcation

When system (1) has a triple zero singularity, similar to the discussion of BT bifurcation, we first give the following lemma.

Lemma 2. *The bases of P and its dual space P^* have the representations $P = \operatorname{span}\Phi$, $\Phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))$, $P^* = \operatorname{span}\Psi$, $\Psi(s) = \operatorname{col}(\psi_1(s), \psi_2(s), \psi_3(s))$, where $\phi_1(\theta) = \phi_1^0 \in \mathbb{R}^n \setminus \{0\}$, $\phi_2(\theta) = \phi_2^0 + \phi_1^0\theta$, $\phi_3(\theta) = \phi_3^0 + \phi_2^0\theta + \phi_1^0\theta^2/2$, $\phi_2^0, \phi_3^0 \in \mathbb{R}^n$ and $\psi_3(s) = \psi_3^0 \in \mathbb{R}^{n*} \setminus \{0\}$, $\psi_2(s) = \psi_2^0 - s\psi_3^0$, $\psi_1(s) = \psi_1^0 - s\psi_2^0 + s^2\psi_3^0/2$, $\psi_1^0, \psi_2^0 \in \mathbb{R}^{n*}$, which satisfy*:

$$\text{(a)}\ \left(A + \sum_{l=1}^m B_l\right)\phi_1^0 = 0, \qquad \text{(b)}\ \left(A + \sum_{l=1}^m B_l\right)\phi_2^0 = \left(\sum_{l=1}^m \tau_l B_l + I\right)\phi_1^0,$$

$$\text{(c)}\ \left(A + \sum_{l=1}^m B_l\right)\phi_3^0 = \left(\sum_{l=1}^m \tau_l B_l + I\right)\phi_2^0 - \frac{1}{2}\sum_{l=1}^m \tau_l^2 B_l \phi_1^0,$$

$$\text{(d)}\ \psi_3^0\left(A + \sum_{l=1}^m B_l\right) = 0, \qquad \text{(e)}\ \psi_2^0\left(A + \sum_{l=1}^m B_l\right) = \psi_3^0\left(\sum_{l=1}^m \tau_l B_l + I\right),$$

$$\text{(f)}\ \psi_1^0\left(A + \sum_{l=1}^m B_l\right) = \psi_2^0\left(\sum_{l=1}^m \tau_l B_l + I\right) - \frac{1}{2}\psi_3^0\sum_{l=1}^m \tau_l^2 B_l,$$

$$\text{(g)}\ \psi_3^0\left(\sum_{l=1}^m \tau_l B_l + I\right)\phi_3^0 - \frac{1}{2}\psi_3^0\sum_{l=1}^m \tau_l^2 B_l\phi_2^0 + \frac{1}{6}\psi_3^0\sum_{l=1}^m \tau_l^3 B_l\phi_1^0 = 1,$$

$$\begin{aligned}\text{(h)}\ &\psi_2^0\left(\sum_{l=1}^m \tau_l B_l + I\right)\phi_3^0 - \frac{1}{2}\psi_2^0\sum_{l=1}^m \tau_l^2 B_l\phi_2^0 + \frac{1}{6}\psi_2^0\sum_{l=1}^m \tau_l^3 B_l\phi_1^0 \\ &- \frac{1}{2}\psi_3^0\sum_{l=1}^m \tau_l^2 B_l\phi_3^0 + \frac{1}{6}\psi_3^0\sum_{l=1}^m \tau_l^3 B_l\phi_2^0 - \frac{1}{24}\psi_3^0\sum_{l=1}^m \tau_l^4 B_l\phi_1^0 = 0,\end{aligned}$$

$$\begin{aligned}\text{(i)}\ &\psi_1^0\left(\sum_{l=1}^m \tau_l B_l + I\right)\phi_3^0 - \frac{1}{2}\psi_1^0\sum_{l=1}^m \tau_l^2 B_l\phi_2^0 + \frac{1}{6}\psi_1^0\sum_{l=1}^m \tau_l^3 B_l\phi_1^0 \\ &- \frac{1}{2}\psi_2^0\sum_{l=1}^m \tau_l^2 B_l\phi_3^0 + \frac{1}{6}\psi_2^0\sum_{l=1}^m \tau_l^3 B_l\phi_2^0 - \frac{1}{24}\psi_2^0\sum_{l=1}^m \tau_l^4 B_l\phi_1^0 \\ &+ \frac{1}{6}\psi_3^0\sum_{l=1}^m \tau_l^3 B_l\phi_3^0 - \frac{1}{24}\psi_3^0\sum_{l=1}^m \tau_l^4 B_l\phi_2^0 + \frac{1}{120}\psi_3^0\sum_{l=1}^m \tau_l^5 B_l\phi_1^0 = 0.\end{aligned}$$

Without considering the factors of coefficient constant, we can determine the unique vectors ϕ_1^0, ϕ_2^0, ϕ_3^0, ψ_1^0, ψ_2^0, ψ_3^0 by (a)–(i).

Note that the proof of this lemma is similar to that of Lemma 1, we omit it here. It is easy to see that $\Phi(\theta)$ satisfies $\dot{\Phi} = \Phi J$, where

$$J = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Defining the Taylor expansion of $\widehat{F}(x_t, \mathbf{P})$ with respect to x_t and $\mathbf{P}$ in system (7) as $\widehat{F}(x_t, \mathbf{P}) = \sum_{w\geqslant 2} \widehat{F}_w(x_t, \mathbf{P})/w!$, we have

$$\begin{aligned} \frac{1}{2}\widehat{F}_2(x_t, \mathbf{P}) = {} & A_1 x(t)\alpha_1 + A_2 x(t)\alpha_2 + A_3 x(t)\alpha_3 \\ & + \sum_{l=1}^{m} \bigl[B_{l1} x(t-\tau_l)\alpha_1 + B_{l2} x(t-\tau_l)\alpha_2 + B_{l3} x(t-\tau_l)\alpha_3\bigr] \\ & + \sum_{i=1}^{m} \sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj} x_i(t-\tau_k) x(t-\tau_j), \end{aligned} \tag{15}$$

where $\tau_0 = 0$ and A_1, A_2, A_3, B_{l1}, B_{l2}, B_{l3}, D_{ikj} are all coefficient matrices. As the discussion above, we also know there is no terms of $O(\mathbf{P}^2)$ in $\widehat{F}_2(x_t, \mathbf{P})$ because $\widehat{F}(0, \mathbf{P}) = 0$ for all $\mathbf{P} \in \mathbb{R}^3$.

Therefore, the normal form for (6) on the center manifold corresponding to the space P takes the form as

$$\dot{z} = Jz + \frac{1}{2} g_2^1(z, 0, \mathbf{P}) + \text{h.o.t.}$$

(see [16] for detail), where

$$g_2^1 = \bigl(I - P_{I,2}^1\bigr) f_2^1, \qquad f_2^1(z, 0, \mathbf{P}) = \Psi(0)\widehat{F}_2(\Phi z, \mathbf{P}).$$

Following [16], we know

$$\begin{aligned} &P_{I,2}^1\bigl(z_1^2, 0, 0\bigr)^{\mathrm{T}} = \bigl(z_1^2, 0, -2z_2^2 - 2z_1z_3\bigr)^{\mathrm{T}}, && P_{I,2}^1(\alpha_i z_1, 0, 0)^{\mathrm{T}} = (\alpha_i z_1, 0, -\alpha_i z_3)^{\mathrm{T}}, \\ &P_{I,2}^1(0, z_1z_2, 0)^{\mathrm{T}} = \bigl(0, z_1z_2, -z_2^2 - z_1z_3\bigr)^{\mathrm{T}}, && P_{I,2}^1\bigl(0, z_1^2, 0\bigr)^{\mathrm{T}} = \bigl(0, z_1^2, -2z_1z_2\bigr)^{\mathrm{T}}, \\ &P_{I,2}^1(0, \alpha_i z_1, 0)^{\mathrm{T}} = (0, \alpha_i z_1, -\alpha_i z_2)^{\mathrm{T}}, && P_{I,2}^1(0, \alpha_i z_2, 0)^{\mathrm{T}} = (0, \alpha_i z_2, -\alpha_i z_3)^{\mathrm{T}}, \end{aligned}$$

and for other p, it has

$$\bigl(I - P_{I,2}^1\bigr)p = \begin{cases} p, & p \in \operatorname{Im}(M_2^1)^c, \\ 0, & p \in \operatorname{Im}(M_2^1), \end{cases}$$

the basis of $\operatorname{Im}(M_2^1)^c$ and $\operatorname{Im}(M_2^1)$ can be seen in [16].

From (15) we have the following expressions, where ϕ_{ji} stands for the ith element of ϕ_j:

$$\begin{aligned}\frac{1}{2}\widehat{F}_2(\Phi z, \mathbf{P}) = {} & A_1\big[\Phi(0)(z_1,z_2,z_3)^{\mathrm{T}}\big]\alpha_1 + A_2\big[\Phi(0)(z_1,z_2,z_3)^{\mathrm{T}}\big]\alpha_2 \\ & + A_3\big[\Phi(0)(z_1,z_2,z_3)^{\mathrm{T}}\big]\alpha_3 + \sum_{l=1}^{m}\big\{B_{l1}\big[\Phi(-\tau_l)(z_1,z_2,z_3)^{\mathrm{T}}\big]\alpha_1 \\ & + B_{l2}\big[\Phi(-\tau_l)(z_1,z_2,z_3)^{\mathrm{T}}\big]\alpha_2 + B_{l3}\big[\Phi(-\tau_l)(z_1,z_2,z_3)^{\mathrm{T}}\big]\alpha_3\big\} \\ & + \sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\big[\big(\phi_{1i}(-\tau_k), \phi_{2i}(-\tau_k) + \phi_{3i}(-\tau_k)\big)(z_1,z_2,z_3)^{\mathrm{T}}\big] \\ & \times \big[\Phi(-\tau_j)(z_1,z_2,z_3)^{\mathrm{T}}\big].\end{aligned}$$

After some computations we get

$$\begin{aligned}&\frac{1}{2}\widehat{F}_2(\Phi z, \mathbf{P}) \\ &\quad = \left[A_1\phi_1(0) + \sum_{l=1}^{m} B_{l1}\phi_1(-\tau_l)\right]\alpha_1 z_1 + \left[A_2\phi_1(0) + \sum_{l=1}^{m} B_{l2}\phi_1(-\tau_l)\right]\alpha_2 z_1 \\ &\qquad + \left[A_3\phi_1(0) + \sum_{l=1}^{m} B_{l3}\phi_1(-\tau_l)\right]\alpha_3 z_1 + \left[A_1\phi_2(0) + \sum_{l=1}^{m} B_{l1}\phi_2(-\tau_l)\right]\alpha_1 z_2 \\ &\qquad + \left[A_2\phi_2(0) + \sum_{l=1}^{m} B_{l2}\phi_2(-\tau_l)\right]\alpha_2 z_2 + \left[A_3\phi_2(0) + \sum_{l=1}^{m} B_{l3}\phi_2(-\tau_l)\right]\alpha_3 z_2 \\ &\qquad + \left[A_1\phi_3(0) + \sum_{l=1}^{m} B_{l1}\phi_3(-\tau_l)\right]\alpha_1 z_3 + \left[A_2\phi_3(0) + \sum_{l=1}^{m} B_{l2}\phi_3(-\tau_l)\right]\alpha_2 z_3 \\ &\qquad + \left[A_3\phi_3(0) + \sum_{l=1}^{m} B_{l3}\phi_3(-\tau_l)\right]\alpha_3 z_3 + \sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\phi_1(-\tau_j)\phi_{1i}(-\tau_k) z_1^2 \\ &\qquad + \sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\big[\phi_1(-\tau_j)\phi_{2i}(-\tau_k) + \phi_2(-\tau_j)\phi_{1i}(-\tau_k)\big] z_1 z_2 \\ &\qquad + \sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\big[\phi_1(-\tau_j)\phi_{3i}(-\tau_k) + \phi_3(-\tau_j)\phi_{1i}(-\tau_k)\big] z_1 z_3 \\ &\qquad + \sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\big[\phi_2(-\tau_j)\phi_{3i}(-\tau_k) + \phi_3(-\tau_j)\phi_{2i}(-\tau_k)\big] z_2 z_3 \\ &\qquad + \sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\phi_2(-\tau_j)\phi_{2i}(-\tau_k) z_2^2 + \sum_{i=1}^{m}\sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\phi_3(-\tau_j)\phi_{3i}(-\tau_k) z_3^2.\end{aligned}$$

Note that $\phi_1(0) = \phi_1(-\tau_l) = \phi_1^0$, $\phi_2(0) = \phi_2^0$, $\phi_2(-\tau_l) = \phi_2^0 - \tau_l\phi_1^0$, $\phi_3(0) = \phi_3^0$, $\phi_3(-\tau_l) = \phi_3^0 - \tau_l\phi_2^0 + \tau_l^2\phi_1^0/2$, $\psi_1(0) = \psi_1^0$, $\psi_2(0) = \psi_2^0$ and $\psi_3(0) = \psi_3^0$, thus we have the following theorem.

Theorem 4. *Let* (H1)–(H3) *hold. Then system* (1) *can be reduced to the following three-dimensional system of ODE on the center manifold at* $(x_t, \mathbf{P}) = (0, 0)$:

$$\begin{aligned} \dot{z}_1 &= z_2, \quad \dot{z}_2 = z_3, \\ \dot{z}_3 &= \eta_1 z_1 + \eta_2 z_2 + \eta_3 z_3 + h_1 z_1^2 + h_2 z_2^2 + h_3 z_1 z_2 + h_4 z_1 z_3 + \text{h.o.t.}, \end{aligned} \tag{16}$$

where

$$\begin{aligned}
\eta_1 &= \psi_3^0\phi_1^0\Bigg\{\Bigg(A_1 + \sum_{l=1}^m B_{l1}\Bigg)\alpha_1 + \Bigg(A_2 + \sum_{l=1}^m B_{l2}\Bigg)\alpha_2 + \Bigg(A_3 + \sum_{l=1}^m B_{l3}\Bigg)\alpha_3\Bigg\}, \\
\eta_2 &= \Bigg\{\psi_2^0\Bigg(A_1 + \sum_{l=1}^m B_{l1}\Bigg)\phi_1^0 + \psi_3^0\Bigg[\Bigg(A_1 + \sum_{l=1}^m B_{l1}\Bigg)\phi_2^0 - \sum_{l=1}^m \tau_l B_{l1}\phi_1^0\Bigg]\Bigg\}\alpha_1 \\
&\quad + \Bigg\{\psi_2^0\Bigg(A_2 + \sum_{l=1}^m B_{l2}\Bigg)\phi_1^0 + \psi_3^0\Bigg[\Bigg(A_2 + \sum_{l=1}^m B_{l2}\Bigg)\phi_2^0 - \sum_{l=1}^m \tau_l B_{l2}\phi_1^0\Bigg]\Bigg\}\alpha_2 \\
&\quad + \Bigg\{\psi_2^0\Bigg(A_3 + \sum_{l=1}^m B_{l3}\Bigg)\phi_1^0 + \psi_3^0\Bigg[\Bigg(A_3 + \sum_{l=1}^m B_{l3}\Bigg)\phi_2^0 - \sum_{l=1}^m \tau_l B_{l3}\phi_1^0\Bigg]\Bigg\}\alpha_3, \\
\eta_3 &= \Bigg\{\psi_1^0\Bigg(A_1 + \sum_{l=1}^m B_{l1}\Bigg)\phi_1^0 + \psi_2^0\Bigg[\Bigg(A_1 + \sum_{l=1}^m B_{l1}\Bigg)\phi_2^0 - \sum_{l=1}^m \tau_l B_{l1}\phi_1^0\Bigg] \\
&\quad + \psi_3^0\Bigg[\Bigg(A_1 + \sum_{l=1}^m B_{l1}\Bigg)\phi_3^0 - \sum_{l=1}^m \tau_l B_{l1}\phi_2^0 + \frac{1}{2}\sum_{l=1}^m \tau_l^2 B_{l1}\phi_1^0\Bigg]\Bigg\}\alpha_1 \\
&\quad + \Bigg\{\psi_1^0\Bigg(A_2 + \sum_{l=1}^m B_{l2}\Bigg)\phi_1^0 + \psi_2^0\Bigg[\Bigg(A_2 + \sum_{l=1}^m B_{l2}\Bigg)\phi_2^0 - \sum_{l=1}^m \tau_l B_{l2}\phi_1^0\Bigg] \\
&\quad + \psi_3^0\Bigg[\Bigg(A_1 + \sum_{l=1}^m B_{l2}\Bigg)\phi_3^0 - \sum_{l=1}^m \tau_l B_{l2}\phi_2^0 + \frac{1}{2}\sum_{l=1}^m \tau_l^2 B_{l2}\phi_1^0\Bigg]\Bigg\}\alpha_2 \\
&\quad + \Bigg\{\psi_1^0\Bigg(A_3 + \sum_{l=1}^m B_{l3}\Bigg)\phi_1^0 + \psi_2^0\Bigg[\Bigg(A_3 + \sum_{l=1}^m B_{l3}\Bigg)\phi_2^0 - \sum_{l=1}^m \tau_l B_{l3}\phi_1^0\Bigg] \\
&\quad + \psi_3^0\Bigg[\Bigg(A_3 + \sum_{l=1}^m B_{l3}\Bigg)\phi_3^0 - \sum_{l=1}^m \tau_l B_{l3}\phi_2^0 + \frac{1}{2}\sum_{l=1}^m \tau_l^2 B_{l3}\phi_1^0\Bigg]\Bigg\}\alpha_3, \\
h_1 &= \psi_3^0\sum_{i=1}^m \sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\phi_1^0\phi_{1i}^0, \\
h_2 &= 2\psi_1^0\sum_{i=1}^m \sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\phi_1^0\phi_{1i}^0 + \psi_2^0\Bigg\{\sum_{i=1}^m \sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\big(\phi_1^0\phi_{2i}^0 + \phi_2^0\phi_{1i}^0\big)
\end{aligned}$$

$$- \sum_{i=1}^{m} \sum_{0\leqslant k\leqslant j\leqslant m} \big[(\tau_k+\tau_j)D_{ikj}\big]\phi_1^0\phi_{1i}^0\Bigg\} + \psi_3^0\Bigg\{\sum_{i=1}^{m} \sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\phi_2^0\phi_{2i}^0$$

$$- \sum_{i=1}^{m} \sum_{0\leqslant k\leqslant j\leqslant m} \tau_j D_{ikj}\phi_1^0\phi_{2i}^0 - \sum_{i=1}^{m} \sum_{0\leqslant k\leqslant j\leqslant m} \tau_k D_{ikj}\phi_{1i}^0\big(\phi_2^0-\tau_j\phi_1^0\big)\Bigg\},$$

$$h_3 = 2\psi_2^0\sum_{i=1}^{m} \sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\phi_1^0\phi_{1i}^0 + \psi_3^0\Bigg\{\sum_{i=1}^{m} \sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\big(\phi_1^0\phi_{2i}^0+\phi_2^0\phi_{1i}^0\big)$$

$$- \sum_{i=1}^{m} \sum_{0\leqslant k\leqslant j\leqslant m} \big[(\tau_k+\tau_j)D_{ikj}\big]\phi_1^0\phi_{1i}^0\Bigg\},$$

$$h_4 = 2\psi_2^0\sum_{i=1}^{m} \sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\phi_1^0\phi_{1i}^0 + \psi_3^0\Bigg\{\sum_{i=1}^{m} \sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\big(\phi_1^0\phi_{2i}^0+\phi_2^0\phi_{1i}^0\big)$$

$$- \sum_{i=1}^{m} \sum_{0\leqslant k\leqslant j\leqslant m} \big[(\tau_k+\tau_j)D_{ikj}\big]\phi_1^0\phi_{1i}^0\Bigg\}$$

$$+ \psi_3^0\Bigg\{\sum_{i=1}^{m} \sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\big(\phi_1^0\phi_{3i}^0+\phi_3^0\phi_{1i}^0\big)$$

$$- \sum_{i=1}^{m} \sum_{0\leqslant k\leqslant j\leqslant m} D_{ikj}\Bigg[\bigg(\tau_j\phi_2^0-\frac{1}{2}\tau_j^2\phi_1^0\bigg)\phi_{1i}^0 + \bigg(\tau_k\phi_{2i}^0-\frac{1}{2}\tau_k^2\phi_{1i}^0\bigg)\phi_1^0\Bigg]\Bigg\}.$$

Let

$$\varPi = \begin{pmatrix} \frac{\partial\eta_1}{\partial\alpha_1} & \frac{\partial\eta_1}{\partial\alpha_2} & \frac{\partial\eta_1}{\partial\alpha_3} \\ \frac{\partial\eta_2}{\partial\alpha_1} & \frac{\partial\eta_2}{\partial\alpha_2} & \frac{\partial\eta_2}{\partial\alpha_3} \\ \frac{\partial\eta_3}{\partial\alpha_1} & \frac{\partial\eta_3}{\partial\alpha_2} & \frac{\partial\eta_3}{\partial\alpha_3} \end{pmatrix}.$$

In addition, we need assume

(H5) $\det \varPi \neq 0$.

Following [1], the bifurcation diagrams of system (16) at the origin are as follows:

(i) system (16) undergoes a transcritical bifurcation when $T = \{(\alpha_1,\alpha_2,\alpha_3)\colon \eta_1 = 0\}$,

(ii) system (16) undergoes a Hopf bifurcation when $H_1 = \{(\alpha_1,\alpha_2,\alpha_3)\colon \eta_3 = -\eta_1/\eta_2,\ \eta_2 < 0\}$,

(iii) system (16) undergoes a Hopf bifurcation at equilibrium $(-\eta_1/h_1, 0, 0)$ when $H_2 = \{(\alpha_1,\alpha_2,\alpha_3)\colon \eta_3 = (h_4/h_1 - h_1/(h_3\eta_1 - h_1\eta_2))\eta_1,\ h_1/(h_3\eta_1 - h_1\eta_2) > 0\}$,

(iv) system (16) undergoes a BT bifurcation when $B = \{(\alpha_1,\alpha_2,\alpha_3)\colon \eta_1 = 0,\ \eta_2 = 0\}$,

(v) system (16) undergoes a zero-Hopf bifurcation when $H_3 = \{(\alpha_1,\alpha_2,\alpha_3)\colon \eta_1 = 0,\ \eta_3 = 0,\ \eta_2 < 0\}$.

5 Application in a recurrent neural network

The authors in [2] and [5] have proposed the following three-node recurrent neural network model with four discrete time delays:

$$\begin{aligned}&\dot{x}_1(t) = -x_1(t) + f\big(x_2(t-\tau_2)\big), \qquad \dot{x}_2(t) = -x_2(t) + f\big(x_3(t-\tau_4)\big),\\ &\dot{x}_3(t) = -x_3(t) + af\big(x_1(t-\tau_1)\big) + bf\big(x_2(t-\tau_3)\big),\end{aligned} \tag{17}$$

where $a, b \in \mathbb{R}$, $\tau_i > 0$, $f(x)$ is a general activation function, which satisfies $f'(0) = 1$.

Let $u_1(t) = x_1(t)$, $u_2(t) = x_2(t-\tau_2)$ and $u_3(t) = x_3(t-\tau_2-\tau_4)$, then system (17) is equivalent to the following system with two delays:

$$\begin{aligned}&\dot{u}_1(t) = -u_1(t) + f\big(u_2(t)\big), \qquad \dot{u}_2(t) = -u_2(t) + f\big(u_3(t)\big),\\ &\dot{u}_3(t) = -u_3(t) + af\big(u_1(t-\tau)\big) + bf\big(u_2(t-\sigma)\big),\end{aligned} \tag{18}$$

where $\tau = \tau_1 + \tau_2 + \tau_4$ and $\sigma = \tau_3 + \tau_4$.

The characteristic equation for linearized system (18) at the equilibrium $(0,0,0)$ is

$$\Delta_0(\lambda) = \lambda^3 + 3\lambda^2 + 3\lambda + 1 - (\lambda+1)b\mathrm{e}^{-\lambda\sigma} - a\mathrm{e}^{-\lambda\tau} = 0. \tag{19}$$

By (19), we have

$$\begin{aligned}&\Delta_0(0) = 1 - b - a, && \Delta_0'(0) = 3 - b + b\sigma + a\tau,\\ &\Delta_0''(0) = 6 + 2b\sigma - b\sigma^2 - a\tau^2, && \Delta_0'''(0) = 6 - 3b\sigma^2 + b\sigma^3 + a\tau^3.\end{aligned}$$

From $\Delta_0(0) = \Delta_0'(0) = 0$ we know that if

$$a = a_0 = \frac{2+\sigma}{\sigma-\tau-1}, \qquad b = b_0 = -\frac{3+\tau}{\sigma-\tau-1},$$

then

$$\Delta_0''(0) = \frac{(3+\tau)\sigma^2 - \tau(2+\tau)\sigma - 6(1+\tau) - 2\tau^2}{\sigma-\tau-1}.$$

Solving $\Delta_0''(0) = 0$, it has

$$\begin{aligned}\sigma_1 &= \frac{\tau(2+\tau) + \sqrt{\tau^4 + 12\tau^3 + 52\tau^2 + 96\tau + 72}}{2(3+\tau)} > 0,\\ \sigma_2 &= \frac{\tau(2+\tau) - \sqrt{\tau^4 + 12\tau^3 + 52\tau^2 + 96\tau + 72}}{2(3+\tau)} < 0.\end{aligned}$$

When $a = a_0$, $b = b_0$ and $\sigma = \sigma_1$, we have

$$\Delta_0'''(0) = \frac{216 + 144\tau^2 + 288\tau + 34\tau^3 + 3\tau^4 - \tau^2\sqrt{\tau^4 + 12\tau^3 + 52\tau^2 + 96\tau + 72}}{-(3+\tau)(6 + \tau^2 + 6\tau - \sqrt{\tau^4 + 12\tau^3 + 52\tau^2 + 96\tau + 72}}.$$

One can verify that $\Delta_0'''(0) > 0$, hence, we get the following lemma.

Lemma 3. *Let*

(I1) $a = a_0$, $b = b_0$, $\sigma \neq \sigma_1$ *and*

(I2) $a = a_1 = (2 + \sigma_1)/(\sigma_1 - \tau - 1)$, $b = b_1 = -(3 + \tau)/(\sigma_1 - \tau - 1)$, $\sigma = \sigma_1$

hold. Then system (18) *has BT* (*triple zero*) *bifurcation at the origin.*

In the following we will analyze the BT and triple zero bifurcations of system (18) at the origin, respectively.

5.1 BT bifurcation of system (18)

In this part, under condition (I1), we take a_0 and b_0 as bifurcation parameters to discuss the BT bifurcation of system (18), and rewrite a and b as $a_0 + \alpha_1$ and $b_0 + \alpha_2$, then we obtain the system

$$\begin{aligned} &\dot{u}_1(t) = -u_1(t) + f\big(u_2(t)\big), \qquad \dot{u}_2(t) = -u_2(t) + f\big(u_3(t)\big), \\ &\dot{u}_3(t) = -u_3(t) + (a_0 + \alpha_1) f\big(u_1(t-\tau)\big) + (b_0 + \alpha_2) f\big(u_2(t-\sigma)\big), \end{aligned} \tag{20}$$

the Taylor expansion of Eq. (20) up to the second order terms is as follows:

$$\dot{U}(t) = AU(t) + B_1 U(t-\tau) + B_2 U(t-\sigma) + \frac{1}{2}\widehat{F}_2(U_t, \mathbf{P}) + \text{h.o.t.},$$

where $U(t) = (u_1(t),\, u_2(t),\, u_3(t))^{\mathrm{T}}$,

$$A = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}, \qquad B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ a_0 & 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b_0 & 0 \end{pmatrix},$$

$\widehat{F}_2(x_t, \mathbf{P})/2 = (f''(0)u_2^2(t)/2,\ f''(0)u_3^2(t)/2,\ \alpha_1 u_1(t-\tau) + \alpha_2 u_2(t-\sigma) + a_0 f''(0) \times u_1^2(t-\tau)/2 + b_0 f''(0) u_2^2(t-\sigma)/2)^{\mathrm{T}}$.

To obtain the normal form of system (20) on its center manifold, by the Lemma 1, we have

$$\Phi(\theta) = \begin{pmatrix} 1 & \theta+1 \\ 1 & \theta+2 \\ 1 & \theta+3 \end{pmatrix}, \qquad \Psi(0) = \begin{pmatrix} \psi_{11} & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} & \psi_{23} \end{pmatrix},$$

where $\gamma = (4+2\sigma)\tau^3 + (30+12\sigma-3\sigma^2)\sigma + (-18\sigma^2+\sigma^3+12\sigma+60)\tau - 6\sigma + 3\sigma^3 - 27\sigma^2 + 42$, and

$$\begin{aligned} \psi_{11} &= \frac{2\gamma(2+\sigma)}{3[(2+\sigma)\tau^2 + (6+2\sigma-\sigma^2)\tau + 6 - 3\sigma^2]}, \\ \psi_{21} &= \frac{-2(2+\sigma)}{6+6\tau+2\tau^2+\tau^2\sigma-3\sigma^2-\sigma^2\tau+2\tau\sigma}, \\ \psi_{22} = \psi_{23} &= \frac{\psi_{21}(-1+\sigma-\tau)}{2+\sigma}, \end{aligned}$$

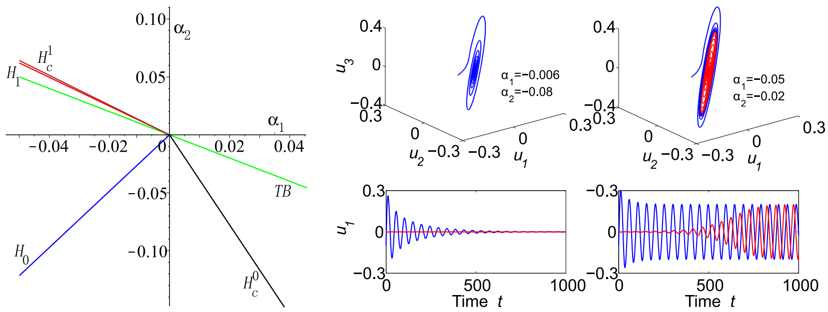

Figure 1. Bifurcation diagram of system (20) and dynamical evolution of the system. The left is the bifurcation diagram, in the middle, a local asymptotically stable equilibrium $(0,0,0)$ is shown with $(\alpha_1,\alpha_2) = (-0.006,-0.08)$. On the right, a stable limit cycle surrounding the equilibrium $(0,0,0)$ is plotted with $(\alpha_1,\alpha_2) = (-0.05,-0.02)$.

$$\psi_{12} = \frac{\psi_{11}\sigma - \psi_{11}\tau + \psi_{21}\tau\sigma + 2\,\psi_{21}\sigma - 3\psi_{21}\tau - \psi_{21}\tau^2 - \psi_{11} - 2\psi_{21}}{2+\sigma},$$
$$\psi_{13} = \frac{-\psi_{11} + \psi_{11}\sigma - \psi_{11}\tau - 2\psi_{21}\tau + \psi_{21}\tau\sigma - \psi_{21} + \psi_{21}\sigma - \psi_{21}\tau^2}{2+\sigma}.$$

By Theorem 3, the delay differential system (20) can be reduced to system (14) on the center manifold at $(u_t,\mathbf{P}) = (0,0)$, where

$$\begin{aligned}
\delta_1 &= \frac{2(1-\sigma+\tau)(\alpha_1+\alpha_2)}{6+6\tau+2\tau^2+\tau^2\sigma-3\sigma^2-\sigma^2\tau+2\tau\sigma},\\
\delta_2 &= -\frac{2(1-\sigma+\tau)}{3(6+6\tau+2\tau^2+\tau^2\sigma-3\sigma^2-\sigma^2\tau+2\tau\sigma)^2}\big[\big(2(2+\sigma)\tau^3\\
&\quad + 3\big(6-\sigma^2+2\sigma\big)\tau^2 + \big(24-12\sigma^2+\sigma^3\big)\tau + 6 - 6\sigma + 3\sigma^3 - 9\sigma^2\big)\alpha_1\\
&\quad + \big((2+\sigma)\tau^3 + 3(2-\sigma-\sigma^2)\tau^2\\
&\quad + 2\big(6-3\sigma^2+\sigma^3-6\sigma\big)\tau - 12\sigma + 12 + 6\sigma^3\big)\alpha_2\big],\\
d_1 &= \frac{(2\tau-3\sigma)f''(0)}{6+6\tau+2\tau^2+\tau^2\sigma-3\sigma^2-\sigma^2\tau+2\tau\sigma},\\
d_2 &= \frac{2f''(0)}{3(6+6\tau+2\tau^2+\tau^2\sigma-3\sigma^2-\sigma^2\tau+2\tau\sigma)^2}\big[2(2+\sigma)\tau^4 + 24(2+\sigma)\tau^3\\
&\quad + \big(-30\sigma^2+168-5\sigma^3+48\sigma\big)\tau^2 + 3\big(-8\sigma+80-36\sigma^2+\sigma^4-2\sigma^3\big)\tau\\
&\quad + 9(\sigma-2)\big(\sigma^3+5\sigma^2+2\sigma-6\big)\big].
\end{aligned}$$

Since $|\partial(\delta_1,\delta_2)/\partial(\alpha_1,\alpha_2)| = 4(1-\sigma+\tau)^3/(6+6\tau+2\tau^2+\tau^2\sigma-3\sigma^2-\sigma^2\tau+2\tau\sigma)^2 \neq 0$, then the map $(\delta_1,\delta_2) \to (\alpha_1,\alpha_2)$ is regular.

Take $\tau = 2$, $\sigma = 0.2$, $f(x) = \tanh(x) + 0.1x^2$, then $a_0 \approx -0.7857142857$, $b_0 \approx 1.785714286$, $\delta_1 \approx 0.2043795620\alpha_1 + 0.2043795620\alpha_2$, $\delta_2 \approx -0.4053776617\alpha_1 + 0.1668851120\alpha_2$, $d_1 \approx 0.01240875912$, $d_2 \approx 0.1551551317$. Thus, the bifurcation diagram in the $\alpha_1\alpha_2$-plane can be obtained as shown on the left of the Fig. 1.

As a verification, we take pair of parameters $(\alpha_1, \alpha_2) = (-0.006, -0.08)$, then we can see that the trivial equilibrium $(0, 0, 0)$ of system (20) is locally asymptotically stable (see the middle of Fig. 1). The trivial equilibrium keeps stable with pair of parameters (α_1, α_2) moving toward the bifurcation curve H_0, and lose its stability when (α_1, α_2) pass through H_0, which leads to a stable limit cycle is bifurcated from the trivial equilibrium. As shown on the right of Fig. 1, system (20) displays a stable periodic orbit when $(\alpha_1, \alpha_2) = (-0.05, -0.02)$.

5.2 Triple zero bifurcation of system (18)

In this part, under condition (I2), we will discuss the triple zero bifurcation of system (18). For computation simplicity, first we rescale system (18) by letting $t = \bar{t}\sigma$, then take a_1, b_1 and σ_1 as bifurcation parameters and write system (18) as

$$\begin{aligned}
\dot{u}_1(t) &= (\sigma_1 + \alpha_3)\bigl[-u_1(t) + f\bigl(u_2(t)\bigr)\bigr],\\
\dot{u}_2(t) &= (\sigma_1 + \alpha_3)\bigl[-u_2(t) + f\bigl(u_3(t)\bigr)\bigr],\\
\dot{u}_3(t) &= (\sigma_1 + \alpha_3)\biggl[-u_3(t) + (a_1 + \alpha_1) f\biggl(u_1\biggl(t - \frac{\tau}{\sigma_1}\biggr)\biggr)\\
&\qquad + (b_1 + \alpha_2) f\bigl(u_2(t-1)\bigr)\biggr],
\end{aligned} \tag{21}$$

the corresponding Taylor expansion of Eq. (21) up to the second order terms is as follows:

$$\dot{U}(t) = AU(t) + B_1 U\biggl(t - \frac{\tau}{\sigma_1}\biggr) + B_2 U(t-1) + \frac{1}{2}\widehat{F}_2(U_t, \mathbf{P}) + \text{h.o.t.},$$

where $U(t) = (u_1(t),\, u_2(t),\, u_3(t))^{\mathrm{T}}$, and

$$A = \sigma_1 \begin{pmatrix} -1 & 1 & 0\\ 0 & -1 & 1\\ 0 & 0 & -1 \end{pmatrix}, \qquad B_1 = \sigma_1 \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ a_1 & 0 & 0 \end{pmatrix}, \qquad B_2 = \sigma_1 \begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & b_1 & 0 \end{pmatrix},$$

$$\begin{aligned}
&\frac{1}{2}\widehat{F}_2(x_t, \mathbf{P})\\
&\quad= \begin{pmatrix}
\alpha_3(-u_1(t) + u_2(t)) + \sigma_1 \frac{f''(0)}{2} u_2^2(t)\\
\alpha_3(-u_2(t) + u_3(t)) + \sigma_1 \frac{f''(0)}{2} u_3^2(t)\\
\sigma_1[\alpha_1 u_1(t - \frac{\tau}{\sigma_1}) + a_1 \frac{f''(0)}{2} u_1^2(t - \frac{\tau}{\sigma_1}) + \alpha_2 u_2(t-1)\\
+ b_1 \frac{f''(0)}{2} u_2^2(t-1)] + \alpha_3[-u_3(t) + a_1 u_1(t - \frac{\tau}{\sigma_1}) + b_1 u_2(t-1)]
\end{pmatrix}^{\mathrm{T}}.
\end{aligned} \tag{22}$$

By using Lemma 2, we can compute

$$\Phi(\theta) = \begin{pmatrix}
1 & \theta - \frac{1}{\sigma_1} & \frac{\sigma_1^2 + 1}{\sigma_1} - \frac{1}{\sigma_1}\theta + \frac{1}{2}\theta^2\\
1 & \theta & 1 + \frac{1}{2}\theta^2\\
1 & \theta + \frac{1}{\sigma_1} & 1 + \frac{1}{\sigma_1}\theta + \frac{1}{2}\theta^2
\end{pmatrix},$$

$$\Psi(0) = \begin{pmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{21} & \omega_{22} & \omega_{23} \\ \frac{\sigma_1+2}{-1+\sigma_1-\tau}\omega_{33} & \omega_{33} & \omega_{33} \end{pmatrix},$$

where

$$\begin{aligned}
\omega_{11} &= \frac{(\sigma_1+2)[2\sigma_1^2\omega_{13} - 2\sigma_1(\tau+1)\omega_{23} + (\tau^2+2\tau+2)\omega_{33}]}{2(-1+\sigma_1-\tau)\sigma_1^2}, \\
\omega_{12} &= \frac{1}{2(-1+\sigma_1-\tau)\sigma_1^2}\big(2\sigma_1(\sigma_1-\tau-1)(\omega_{13}\sigma_1+\omega_{23}) \\
&\quad - \big(\sigma_1^2\tau - \sigma_1\tau^2 + 3\sigma_1^2 - 2\sigma_1\tau - 2\tau^2 - 6\tau - 6\big)\omega_{33}\big), \\
\omega_{21} &= \frac{(\sigma_1+2)(\omega_{23}\sigma_1 - \omega_{33}\tau - \omega_{33})}{(-1+\sigma_1-\tau)\sigma_1}, \qquad \omega_{22} = \frac{\omega_{23}\sigma_1 + \omega_3 3}{\sigma_1}, \\
\omega_{13} &= -\frac{3(-1+\sigma_1-\tau)}{40[(3+\tau)\sigma_1^3 - (\tau^3+3\tau^2+6\tau+6)\sigma_1 - 2(\tau_0^3+3\tau^2+6\tau-6)]^3} \\
&\quad \times \big[\tau^7(\sigma_1+2)^2(\tau+8) - 2\tau^3(\sigma_1+2)\big(\big(-22\sigma_1^3 - 40\sigma_1^2 - 18\sigma_1 - 36\big)\tau^3 \\
&\quad + \big(65\sigma_1^4 - 16\sigma_1^3 - 120\sigma_1^2 - 48\sigma_1 - 96\big)\tau^2 + \big(-442\sigma_1^5 - 385\sigma_1^4 + 170\sigma_1^3 \\
&\quad - 120\sigma_1^2 - 60\sigma_1 - 120\big)\tau + 780\sigma_1^6 - 2052\sigma_1^5 - 3480\sigma_1^4 + 240\sigma_1^3 - 480\sigma_1^2\big) \\
&\quad + \sigma_1^2\big(\big(761\sigma_1^6 - 8040\sigma_1^5 - 12180\sigma_1^4 + 29520\sigma_1^3 + 34560\sigma_1^2 + 3840\sigma_1 \\
&\quad + 8640\big)\tau^2 + \big(4566\sigma_1^6 - 10800\sigma_1^5 - 32064\sigma_1^4 + 24192\sigma_1^3 + 37440\sigma_1^2 \\
&\quad + 3840\sigma_1 + 11520\big)\tau + 6849\sigma_1^6 - 2160\sigma_1^5 \\
&\quad - 25128\sigma_1^4 + 5184\sigma_1^3 + 14400\sigma_1^2 + 5760\big)\big], \\
\omega_{23} &= -\frac{3\sigma_1(-1+\sigma_1-\tau)}{2[(3+\tau)\sigma_1^3 - (\tau^3+3\tau^2+6\tau+6)\sigma_1 - 2(\tau_0^3+3\tau^2+6\tau-6)]^2} \\
&\quad \times \big[\sigma_1^2\big(13\sigma_1^2\tau + 39\sigma_1^2 - 24\sigma_1\tau - 72\tau - 72\big) \\
&\quad - (\sigma_1+2)\big(12\sigma_1^2\tau^2 + \tau^4 + 4\tau^3 + 12\tau^2 + 24\tau + 24\big)\big], \\
\omega_{33} &= \frac{-6(-1+\sigma_1-\tau)\sigma_1^2}{(3+\tau)\sigma_1^3 - (\tau^3+3\tau^2+6\tau+6)\sigma_1 - 2(\tau^3+3\tau^2+6\tau+6)}.
\end{aligned}$$

By (22), we know the expressions of coefficient matrices in (15), and A_3, B_{11}, B_{12}, B_{13}, B_{23}, D_{200}, D_{300}, D_{111} and D_{222} are zero matrices. Furthermore, by Theorem 4, the delay differential system (21) can be reduced to system (16) on the center manifold at $(u_t, \mathbf{P}) = (0, 0)$, where

$$\begin{aligned}
\eta_1 &= \omega_{33}\sigma_1(\alpha_1+\alpha_2), \\
\eta_2 &= -\frac{\omega_{33}}{(1-\sigma_1+\tau)\sigma_1}\big[\sigma_1(1-\sigma_1+\tau)(\tau\alpha_1+\alpha_2\sigma_1+\alpha_1) - \alpha_3(2\tau-3\sigma_1)\big], \\
\eta_3 &= \frac{1}{2(1-\sigma_1+\tau)\sigma_1^2}\big[\sigma_1(1-\sigma_1+\tau)\big(\big(2\sigma_1^3+\tau^2+2\tau+2\sigma_1\big)\omega_{33} \\
&\quad - 2\sigma_1(\tau+1)\omega_{23} + 2\omega_{13}\sigma_1^2\big)\alpha_1 + (\tau-\sigma_1+1)\sigma_1^3(2\omega_{13} - 2\omega_{23} + 3\omega_{33})\alpha_2
\end{aligned}$$

$$
\begin{aligned}
&\quad + \big(2\sigma_1(\tau-\sigma_1+1)(\omega_{21}+\omega_{22}) + 2\omega_{23}\sigma_1(\tau-\sigma_1) \\
&\quad + \big(\tau\sigma_1^2 - \tau^2\sigma_1 - 2\tau^2 - 2\tau\sigma_1 + 3\sigma_1^2 - 4\tau\big)\omega_{33}\big)\alpha_3\big], \\
h_1 &= \frac{\omega_{33}\sigma_1(2\tau-3\sigma_1)f''(0)}{1-\sigma_1+\tau}, \\
h_2 &= -\frac{1}{\sigma_1}\big((\tau\sigma_1+2\tau+3\sigma_1+1)\omega_{33} \\
&\quad - 2\sigma_1(\omega_{11}\sigma_1+\omega_{12}\sigma_1+\omega_{13}\sigma_1+\omega_{22}+2\omega_{23})\big)f''(0), \\
h_3 &= 2(\omega_{21}\sigma_1+\omega_{22}\sigma_1+\omega_{23}\sigma_1+3\omega_{33})f''(0), \\
h_4 &= \frac{-f''(0)}{\sigma_1(1-\sigma_1+\tau)}\big[\omega_{33}\big(2\sigma_1^4+\tau^2\sigma_1-5\tau\sigma_1^2+8\sigma_1^3+2\tau^2+2\tau\sigma_1-5\sigma_1^2 \\
&\quad + 4\tau + 4\sigma_1\big) - 2\sigma_1(\omega_{11}\sigma_1+\omega_{12}\sigma_1+\omega_{13}\sigma_1+\omega_{22})(1-\sigma_1+\tau) \\
&\quad - 4\omega_{23}\big(\tau\sigma_1-\sigma_1^2+1\big)\big].
\end{aligned}
$$

Take $\tau = 0.2$, $f(x) = \tanh(x) + x^2/2$, then $a_1 \approx 9.451604291$, $b_1 \approx -8.451604291$, $\sigma_1 \approx 1.578626340$, $f''(0) = 1$, $\eta_1 \approx 0.6554453042\alpha_1 + 0.6554453042\alpha_2$, $\eta_2 \approx -0.6554453042\alpha_2 - 0.4982397323\alpha_1 + 3.011924622\alpha_3$, $\eta_3 \approx 0.5239069738\alpha_1 - 0.1559415893\alpha_2 - 0.156702101\alpha_3$, $h_1 \approx 7.505900258$, $h_2 \approx -12.39030109$, $h_3 \approx 8.728384667$, $h_4 \approx 14.89038763$.

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References

1. S.A. Campbell, Y. Yuan, Zero singularities of codimension two and three in delay differential equations, *Nonlinearity*, **21**(11):2671–2691, 2008.
2. Y. Ding, W. Jiang, P. Yu, Bifurcation analysis in a recurrent neural network model with delays, *Commun. Nonlinear Sci. Numer. Simul.*, **18**(2):351–372, 2013.
3. G. Fan, S.A. Campbell, G.S.K. Wolkowicz, H.P. Zhu, The bifurcation study of 1:2 resonance in a delayed system of two coupled neurons, *J. Dyn. Differ. Equations*, **25**(1):193–216, 2013.
4. T. Faria, L.T. Magalhaes, Normal forms for retarded functional differential equations and applications to Bogdanov–Takens singularity, *J. Differ. Equations*, **122**(2):201–224, 1995.
5. C.H. Feng, C. O'Reilly, R. Plamondon, Permanent oscillations in a 3-node recurrent neural network model, *Neurocomputing*, **74**(1-3):274–283, 2010.
6. E. Freire, E. Gamero, A.J. Rodriguezluts, A. Algaba, A note on the triple-zero linear degeneracy: Normal forms, dynamical and bifurcation behaviors of an unfolding, *Int. J. Bifurcation Chaos Appl. Sci. Eng.*, **12**(12):2799–2820, 2002.
7. J.K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.
8. J.K. Hale, S.M. Lunel, *Introduction to Functional Differential Equations*, Springer-Verlag, New York, 1973.

9. X. He, C. Li, T. Huang, C. Li, Bogdanov–Takens singularity in tri-neuron network with time delay, *IEEE Trans. Neural Netw. Learn. Syst.*, **24**(6):1001–1007, 2013.
10. X. He, C. Li, Y. Shu, Bogdanov–Takens bifurcation in a single inertial neuron model with delay, *Neurocomputing*, **89**:193–201, 2012.
11. X. He, C. Li, Y. Shu, Triple-zero bifurcation in van der Pol's oscillator with delayed feedback, *Commun. Nonlinear Sci. Numer. Simul.*, **17**(12):5229–5239, 2012.
12. X. He, C.D. Li, T.W. Huang, C.J. Li, Codimension two bifurcation in a delayed neural network with unidirectional coupling, *Nonlinear Anal., Real World Appl.*, **14**(2):1191–1202, 2013.
13. J. Jiang, Y. Song, Bogdanov–Takens bifurcation in an oscillator with negative damping and delayed position feedback, *Appl. Math. Modelling*, **37**(16–17):8091–8105, 2013.
14. W. Jiang, Y. Yuan, Bogdanov–Takens singularity in van der Pol's oscillator with delayed feedback, *Physica D*, **227**(2):149–161, 2007.
15. Z. Liu, R. Yuan, Bifurcations in predator–prey systems with nonmonotonic functional response, *Nonlinear Anal., Real World Appl.*, **6**(1):187–205, 2005.
16. Z.Q. Qiao, X.B. Liu, D.M. Zhu, Bifurcation in delay differential systems with triple-zero singularity, *Chin. Ann. Math., Ser. A*, **31**(1):59–70, 2010.
17. J. Wang, W. Jiang, Bogdanov–Takens singularity in the comprehensive national power model with time delays, *J. Appl. Anal. Comput.*, **3**(1):81–94, 2013.
18. J. Xia, Z. Liu, R. Yuan, S. Ruan, The effects of harvesting and time delay on predator–prey systems with Holling type II functional response, *SIAM J. Appl. Math.*, **70**(4):1178–1200, 2009.
19. D. Xiao, S. Ruan, Multiple bifurcations in a delayed predator–prey system with nonmonotonic functional response, *J. Differ. Equations*, **176**(2):494–510, 2001.
20. Y.X. Xu, M.Y. Huang, Homoclinic orbits and Hopf bifurcations in delay differential systems with T–B singularity, *J. Differ. Equations*, **244**(3):582– 598, 2008.
21. P. Yu, Y. Yuan, The simplest normal forms associated with a triple zero eigenvalue of indices one and two, *Nonlinear Anal., Theory Methods Appl.*, **47**(2):1105–1116, 2001.
22. Y. Yuan, J. Wei, Singularity analysis on a planar system with multiple delays, *J. Dyn. Differ. Equations*, **19**(2):437–456, 2007.
23. J.F. Zhang, W.T. Li, X.P. Yan, Multiple bifurcations in a delayed predator–prey diffusion system with a functional response, *Nonlinear Anal., Real World Appl.*, **11**(4):2708–2725, 2010.