The Method of Prime Costs Determination of the Model Row Goods

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Received: 08.02.2008 **Revised:** 23.06.2008 **Published online:** 28.08.2008

Abstract. The concept of the model row goods is introduced. These are the interchangeable goods differing by quality and price. Cognacs of various vintage years produced on one cognac factory are a typical example of such goods. For the indicated kind of the goods the method of the cost price determination of the goods of competitors is worked out and realized. The initial information for determination is the data on the prices of the goods and sales volumes.

Keywords: model row of goods, determination of prime costs, inverse problem to optimization.

1 Introduction

In [1, 2] the class of the goods convenient for economic research is introduced. This class of the goods has received a name the “model row goods”. The stated class is made of the interchangeable goods distinguished by the price and quality. A typical example of such goods is cognacs of various vintage years produced on the same cognac factory. The quality in this case is determined by the vintage year. The price of cognac too depends on its age.

For the model row goods the problem is solved [1, 2] which in this work we shall call a “direct problem”. This problem is formulated as follows. It is necessary to find such change of the prices on the goods being sold that will provide with both profit increase and profitability growth. Thus cost prices of the goods, the buyer prices and volumes of the sold goods are considered known.

In the current work the problem which we name the “inverse problem” is solved. This problem consists in the cost price definition of the competitors goods. Thus the prices of the goods for the buyer and volumes of the sold goods are considered known.

In the following paragraph we shall cite the necessary facts from the direct problem theory. In the paragraph 3 the algorithm of the solution of the inverse problem is given.

2 Approximation of sales volumes dependences

Let a model row consist of the k goods, where each subsequent good is of a better quality than the previous one.

Let:

c_i – be the cost price of the i-th goods unit,

p_i – be the buyer price for the i-th goods unit,

Q_i – be the quantity of the sold i-th goods units for the certain period (month, year, ...), $1 \leq i \leq k$.

We assume, that cost prices c_i are constants, and sales volumes Q_i are functions of the prices p_j. More precisely: $Q_i = Q_i(p_{i-1}, p_i, p_{i+1})$, $2 \leq i \leq k-1$, $Q_1 = Q_1(p_1, p_2)$, $Q_k = Q_k(p_{k-1}, p_k)$, i.e. the sales volume of goods Q_i depends only on the goods prices of almost the same quality.

Let's explain validity of the made assumptions on an example of cognac sale. The sales volume of 5-year-old cognac depends on the very price for 5-year-old cognac, and also on the prices for 4- and 6-year-old cognac. The prices of other cognacs practically do not influence the sales of 5-year-old cognac. For example, 15-year-old cognac is too expensive from the point of view of the consumer of 5-year-old cognac. And three-year-old cognac in his opinion has poor quality.

Certainly, external factors have an effect on sales volumes also: a parity of euro/dollar, the price for petroleum etc. We assume that, in the considered model, all these factors are fixed.

Let us assume that at present the prices of the goods are equal $p_i^{(0)}$, and the sales volumes appropriate to them are equal $Q_i^{(0)}$, $1 \leq i \leq k$.

Some obvious economic and mathematical reasons allow to describe functions Q_i rather precisely. For example, let's consider function $Q_1 = Q_1(p_1, p_2)$ – a sales volume of the 1-st kind goods.

Let's fix the value $p_1 = p_i^{(0)}$. As result we have function of one variable p_2: $Q_1(p_1 = p_i^{(0)}, p_2)$. We shall consider this function on the closed interval $p_1^{(0)} \leq p_2 \leq p_3^{(0)}$. That is the situation when the price of the 1-st kind goods does not vary, and the price of the 2-nd kind goods has changed. One value of this function is known: $Q_1(p_i^{(0)}, p_2 = p_2^{(0)}) = Q_1^{(0)}$. For the price $p_2 = p_1^{(0)}$ value of function too can be named: it is equal to zero. Indeed, nobody buys less qualitative goods if the prices are equal.

Let's find still another value: $Q_1(p_i^{(0)}, p_2 = p_3^{(0)})$, i.e. when the price of 2-nd kind production will be equal to the 3-rd kind price. What will happen to the sales of the 1-st kind goods? In this case there will be no buyers of less quality kind of the goods (2-nd). Then consumers of the 2-nd kind of production will break into three categories. The first (α) – will proceed on consumption of the 1-st kind of production (of the less quality and cheaper). The second (β) – will buy the 3-rd kind of production. The third (γ) will leave the market of the given manufacturer. Coefficients α, β, γ satisfy to conditions:

$$\alpha \geq 0, \quad \beta \geq 0, \quad \gamma \geq 0, \quad \alpha + \beta + \gamma = 1.$$

Concrete values of these coefficients depend on production and a situation in the market. To the market of cognacs these coefficients have the following values: $\alpha = 0.6$, $\beta = 0.3$, $\gamma = 0.1$. They are received as a result of the analysis of sales statistics. Cases when on sale there is no cognac of some kind (for example, 4-year-old cognac) are studied. Then sales volumes of adjacent grades of 3- and 5-year-old cognacs grow. This gain enables to determine coefficients α, β.

Let's assume, that function $Q_1(p_i^{(0)}, p_2)$ is differentiable. Then it is easy to see, that the derivative of this function in points $p_2 = p_1^{(0)}$, $p_2 = p_3^{(0)}$ equals zero. It is also obvious, that $Q_1(p_i^{(0)}, p_2)$ is strictly increasing function on the closed interval $p_1^{(0)} \le p_2 \le p_3^{(0)}$.

Thus, we have the following information on restriction of the function $Q_1(p_1, p_2)$ on a straight line $p_1 = p_1^{(0)}$.

$$
\begin{aligned}
&Q_1(p_1^{(0)}, p_2 = p_2^{(0)}) = Q_1^{(0)}, \quad Q_1(p_1^{(0)}, p_2 = p_1^{(0)}) = 0,\\
&Q_1(p_1^{(0)}, p_2 = p_3^{(0)}) = Q_1^{(0)} + \alpha Q_2^{(0)},\\
&\frac{\partial Q_1}{\partial p_2}(p_1^{(0)}, p_2 = p_1^{(0)}) = 0, \quad \frac{\partial Q_1}{\partial p_2}(p_1^{(0)}, p_2 = p_3^{(0)}) = 0,
\end{aligned}
$$

$Q_1 = Q_1(p_1^{(0)}, p_2)$ – strictly increasing function on the closed interval $p_1^{(0)} \le p_2 \le p_3^{(0)}$.

Certainly, this information does not determine function uniquely, but allows to approximate function precisely enough.

Let's note, that in a similar situation, in the economic theory logistic functions are used. There the argument p_2 varies on all numerical axis. Therefore exponents participate in function record. In a considered situation the argument varies on the closed interval. Therefore the use of trigonometric functions appeared to be more convenient. As a result the following approximation is chosen:

$$
Q_1(p_1^{(0)}, p_2) = \frac{1}{2}\big(Q_1^{(0)} + \alpha Q_2^{(0)}\big)\big(1 - \cos(\pi t)\big), \tag{1}
$$

where

$$
x^{(0)} = \frac{p_2^{(0)} - p_1^{(0)}}{p_3^{(0)} - p_1^{(0)}}, \quad t^{(0)} = \frac{1}{\pi}\arccos\left(1 - 2\frac{Q_1^{(0)}}{Q_1^{(0)} + \alpha Q_2^{(0)}}\right), \quad x = \frac{p_2 - p_1^{(0)}}{p_3^{(0)} - p_1^{(0)}}
$$

and

$$
\begin{aligned}
&t = x, && \text{if } t^{(0)} = x^{(0)},\\
&t = \frac{(1-b)x}{x-b}, \ \text{ where } b = \frac{(1-t^{(0)})x^{(0)}}{x^{(0)} - t^{(0)}}, && \text{if } t^{(0)} \ne x^{(0)}.
\end{aligned}
$$

The indicated representation of function Q_1 enables to calculate a derivative $\frac{\partial Q_1}{\partial p_2}$ in a point $(p_1^{(0)}, p_2^{(0)})$. Similar reasoning allows to approximate restriction of function $Q_1(p_1, p_2)$ on the straight line $p_2 = p_2^{(0)}$. This enables to calculate a derivative $\frac{\partial Q_1}{\partial p_1}$ in a point $(p_1^{(0)}, p_2^{(0)})$. The same reasons are applied to other functions Q_i.

Let:

$C(p_1, p_2, \dots, p_k)$ – be the general cost price of manufacture of a model row goods,
$P(p_1, p_2, \dots, p_k)$ – be the general profit received as a result of realization of model row goods,
$r(p_1, p_2, \dots, p_n)$ – be profitability of manufacture of model row goods.

According to the assumptions these functions are related by the following formulas.

$$
\begin{aligned}
C(p_1, p_2, \dots, p_k) &= \sum_{i=1}^{n} c_i Q_i(p_1, p_2, \dots, p_k), \\
P(p_1, p_2, \dots, p_k) &= \sum_{i=1}^{n} (p_i - c_i) Q_i(p_1, p_2, \dots, p_k), \\
r(p_1, p_2, \dots, p_k) &= \frac{P(p_1, p_2, \dots, p_k)}{C(p_1, p_2, \dots, p_k)}.
\end{aligned}
\tag{2}
$$

Let cost prices of the goods c_i be known. Then the values of functions C, P, r in a point $M_0 = (p_1 = p_1^{(0)}, p_2 = p_2^{(0)}, \dots, p_k = p_k^{(0)})$ are also known. Moreover, due to the explicit form of (1), partial derivatives of these functions in point M_0 are also known. Hence, in this point it is possible to find gradients of functions P, r.

According to the sense of a gradient, any small enough change of arguments of function in half-space from point M_0 results in increase of function value. Hence, change of variables $p_1, p_2, \dots, p_n$ in half-space in which specifies $\operatorname{grad} P$, results in increase of the profit. Change of variables $p_1, p_2, \dots, p_n$ in half-space in which specifies $\operatorname{grad} r$, results in increase of profitability. Therefore, small enough change of the prices $p_1, p_2, \dots, p_n$ in the dihedral angle (crossing of two indicated half-spaces) from point M_0 results both in increase of the profit and in increase of profitability.

The solution of a direct problem, that is of optimization of the prices of the model row goods, is based on the above- mentioned reasoning $[1, 2]$.

In this work we solve an inverse problem – of determination of the cost prices of the model row goods.

3 Determination of cost prices for the model row goods made by competitors

Let's start with the following assumptions. The manufacturer of the model row goods fixes the "optimum" prices for the goods. More precisely, the criterion of an optimality of the prices is formulated as follows: gradients of functions of the profit and profitability $\operatorname{grad} P$ and $\operatorname{grad} r$ are opposite directed.

Now we assume, that cost prices c_i in formulas (2) are unknown to us. Then for a point $M_0(p_1 = p_1^{(0)}, p_2 = p_2^{(0)}, \dots, p_k = p_k^{(0)})$ we shall introduce function

$g(c_1, c_2, \ldots, c_k) = \cos\gamma$, where γ is an angle between gradients $\operatorname{grad} P$, $\operatorname{grad} r$. It's known [1], that it is possible to calculate $\cos\gamma$ through scalar product:

$$g(c_1, c_2, \ldots, c_k) = \cos\gamma = \frac{\big(\operatorname{grad} P(M_0),\ \operatorname{grad} r(M_0)\big)}{|\operatorname{grad} P(M_0)||\operatorname{grad} r(M_0)|}. \tag{3}$$

Thus, for definition of cost prices c_i, $1 \le i \le k$, we have the equation

$$g(c_1, c_2, \ldots, c_k) = -1, \tag{4}$$

where function g is determined in (3).

Probably, it's worth reminding, that $\cos 180^\circ = -1$.

Inverse problems, as a rule, are incorrect (ill-posed) [3,4]. In full measure it concerns a problem (4). As a rule, there is no exact solution to an incorrect problem, in particular to the equations (4). It is necessary to consider so-called the quasi-solution.

For the equation (4) pseudo-solution is the point of a minimum of function $g(c_1, c_2, \ldots, c_k)$. The problem of defining a point of a minimum is unstable. To regularize instability the additional information is usually involved in. This additional information should provide uniqueness of the decision and a belonging of the solution to compact set. For this purpose the number of required parameters is usually reduced and the limits of their change are bounded. We shall assume, that cost prices c_i linearly depend on number i. For cognacs manufacture this assumption is quite justified.

In the cost price of cognacs there is a part, identical to all sorts. It reflects cost of grapes and some other expenses. The second part of the cost price is directly proportional to the vintage of cognac. These are expenses for keeping cognac in oak barrels. The following formula therefore is true: the cost price

$$c_i = a + ib, \tag{5}$$

where a, b are some positive constants, and i is quantity of vintage years of cognac. The restriction is obvious:

$$0 \le c_i \le p_i^{(0)}, \tag{6}$$

where $p_i^{(0)}$ – the price of the buyer for i-th sort of cognac.

Thus, it is necessary to determine two parameters a and b from (5). Thus restriction (6) should be made.

Let's consider a specific example. In Table 1 we are given the prices for wholesale buyers and sales volumes for one of cognac factories of Armenia for concrete year of the current century.

For these data approximations of functions $Q_i(p_{i-1}, p_i, p_{i+1})$, $i = 4, 5$, $Q_3(p_3, p_4)$, $Q_6(p_5, p_6)$ under formulas of a kind (1) are made. With the help of these functions partial derivatives $\frac{\partial Q_i}{\partial p_j}(M_0)$, where $M_0(p_3 = p_3^{(0)}, \ldots, p_6 = p_6^{(0)}) = M_0(2.72, 3.11, 3.56, 4.17)$, are calculated (Table 2).

Table 1. Prices and sales quantities

Vintage (in years)	3	4	5	6
Price (USA dollars)	2.72	3.11	3.56	4.17
Sales quantity (thousand of bottles)	15	9	13	6

Table 2. Quotient derivatives $\frac{\partial Q_i}{\partial p_j}(M_0),\ i,j = 3,4,5,6\ (thous.\, bottles/dollar)$

p_3	$-60{,}415$	$30{,}207$	–	–
p_4	$30{,}540$	$-38{,}469$	$9{,}6269$	–
p_5	–	$25{,}358$	$-27{,}0070$	$5{,}3375$
p_6	–	–	$6{,}4153$	$-6{,}7299$

On these partial derivatives the function from (3) is made

$$g\big(c_3(a,b), c_4(a,b), c_5(a,b), c_6(a,b)\big) = g(a+3b, a+4b, a+5b, a+6b) = \widetilde{g}(a,b). \quad (7)$$

For function (7) the gradiential method finds a point of a minimum. It is numerically established, that there is one point of a minimum. For this purpose, descent began from several different, enough far points. In result values $a = 1,2,\ b = 0,33$ and cost prices of cognacs are received (Table 3).

Table 3. Prices, sales quantities and costs

Vintage (in years)	3	4	5	6
Price (USA dollars)	2.72	3.11	3.56	4.17
Sales quantity (thousand of bottles)	15	9	13	6
Cost (USA dollars)	2.01	2.34	2.47	2.70

Thus, the information on the prices and sales volumes (Table 1) can be added with data on the cost price of the goods (Table 3).

The obtained result is used on several cognac factories of Armenia and Russia.

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