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Some new fixed point results in rectangular metric spaces with an application to fractional-order functional differential equations

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Abstract. In this paper, we establish some new fixed point theorems for generalized ϕ–ψ-contractive mappings satisfying an admissibility-type condition in a Hausdorff rectangular metric space with the help of C-functions. In this process, we rectify the proof of Theorem 3.2 due to Budhia et al. [New fixed point results in rectangular metric space and application to fractional calculus, *Tbil. Math. J.*, 10(1):91–104, 2017]. Some examples are given to illustrate the theorems. Finally, we apply our result (Corollary 3.6) to establish the existence of a solution for an initial value problem of a fractional-order functional differential equation with infinite delay.

Keywords: fixed point, rectangular metric, nonlinear fractional differential equation.

1 Introduction

One of the fundamental results in the evolution of the field of fixed point theory is the Banach contraction principle [8]. It has been generalized and extended in various directions. Among those directions, we came across some new types of metric spaces in literature as the one introduced by Branciari [10]. He gave the concept of a generalized

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metric space (or rectangular metric space or RMS) in which the triangle inequality is replaced with a weaker assumption called quadrilateral inequality, and an analogue of the Banach contraction principle is proved. Then after, many authors initiated and studied fixed point theory in such spaces. For more details about fixed point theory in generalizedmetric spaces, we refer the reader to [3–6, 23, 33, 35–37]. In principle, rectangular metric spaces can lack the Hausdorffness separation (see Example 3 and examples given in [23, 33]), although it is not useful for our theory as Hausdorffness separation plays an important role in Theorem 1 and its corollaries.

In 2012, Samet et al. [34] introduced the concept of α–ψ-contractive mapping, which is interesting since it does not require the contractive conditions to hold for every pair of points in the domain unlike Banach contraction principle. It also includes the case of discontinuous mappings. Because of these reasons, there is massive growth in the literature dealing with fixed point problems via α-admissible mappings (cf. [1, 2, 17, 24, 32]).

Most recently, two different generalizations of α-admissible mapping were given in which the author Ansari [2] used the idea of C-class functions, whereas Budhia et al. [11] used a rectangular metric. Following the ideas from [11] and [2], we provide new fixed point results in generalized metric spaces, which are also utilized to establish the existence of a solution for an initial value problem of a fractional-order functional differential equation with infinite delay.

2 Mathematical preliminaries

In this section, we build up the base for our main results.

Definition 1. (See [10].) Let $M \neq \emptyset$ be a set. A generalized metric (rectangular metric) is a function $d : M \times M \to [0, \infty)$, where the following conditions are fulfilled for all $a, b, x, y \in M$ with $x \neq y$ and $x, y \notin \{a, b\}$:

(i) $d(a, b) = 0$ if and only if $a = b$;
(ii) $d(a, b) = d(b, a)$;
(iii) $d(a, b) \leqslant d(a, x) + d(x, y) + d(y, b)$ (quadrilateral inequality).

The pair (M, d) is named as a generalized metric space (a rectangular metric space or RMS).

Definition 2. (See [10].) Let (M, d) be a generalized metric space, and let $\{s_n\}$ be a sequence in M.

(i) If $d(s_n, s) \to 0$ as $n \to \infty$, then $\{s_n\}$ is called (g.m.s) convergent to a limit s.
(ii) If for every $\epsilon > 0$, there exists $n(\epsilon) \in \mathbb{N}$ such that $d(s_i, s_j) < \epsilon$ for all $i > j \geqslant n(\epsilon)$, then $\{s_n\}$ is called a Cauchy sequence in RMS.
(iii) A rectangular metric space (M, d) is called complete if every (g.m.s) Cauchy sequence is (g.m.s) convergent.

Definition 3. (See [32].) Let $\alpha, \eta : M \times M \to [0, \infty)$ be two mappings. A map $T : M \to M$ is said to be α-admissible with respect to η if $\alpha(Tu_1, Tu_2) \geqslant \eta(Tu_1, Tu_2)$ whenever $\alpha(u_1, u_2) \geqslant \eta(u_1, u_2)$ for all $u_1, u_2 \in M$. If $\eta(u_1, u_2) = 1$ for all $u_1, u_2 \in M$, then T is called an α-admissible mapping.

Definition 4. (See [2].) A C-function $F : [0, \infty)^2 \to \mathbb{R}$ is a continuous function such that for all $s_1, s_2 \in [0, \infty)$:

(i) $F(s_1, s_2) \leqslant s_1$;
(ii) $F(s_1, s_2) = s_1$ implies that either $s_1 = 0$ or $s_2 = 0$.

The letter $\mathcal{C}$ will denote the class of all C-functions.

Example 1. (See [2].) The following are C-functions:

(E1) $F(s_1, s_2) = s_1 - s_2$;
(E2) $F(s_1, s_2) = ms_1$, where $0 < m < 1$;
(E3) $F(s_1, s_2) = s_1 - s_2/(k + s_2)$;
(E4) $F(s_1, s_2) = \log(s_2 + a^{s_1})/(1 + s_2)$ for some $a > 1$;
(E5) $F(s_1, s_2) = s_1 - \phi(s_1)$, where $\phi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\phi(s) = 0$ if and only if $s = 0$;
(E6) $F(s_1, s_2) = s_1 k(s_1, s_2)$, where $k : [0, \infty) \times [0, \infty) \to [0, \infty)$ is a continuous function such that $k(s_1, s_2) < 1$ for all $s_1, s_2 > 0$;
(E7) $F(s_1, s_2) = \psi(s_1)$, where $\psi : [0, \infty) \to [0, \infty)$ is a continuous function such that $\psi(0) = 0$ and $\psi(s) < s$ for $s > 0$.

Definition 5. (See [25].) A nondecreasing continuous function $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if $\psi(s) = 0$ if and only if $s = 0$.

Remark 1. We denote by Ψ the class of altering distance functions.

The following lemma is useful for the rest and its proof is classical. We omit it.

Lemma 1. *Let (M, d) be a complete RMS and $\{s_n\}$ be a sequence in M such that $\lim_{n\to\infty} d(s_n, s_{n+1}) = 0 = \lim_{n\to\infty} d(s_n, s_{n+2})$ and $s_n \neq s_m$ for all positive integers $n \neq m$. If $\{s_n\}$ is not a Cauchy sequence, then there exist an $\varepsilon > 0$ and sequences $\{m(k)\}$ and $\{n(k)\}$ in $\mathbb{N}$ with $m(k) > n(k) > k$ with $d(s_{m(k)}, s_{n(k)}) \geqslant \varepsilon$, $d(s_{m(k)-1}, s_{n(k)}) < \varepsilon$ so that the following hold*:

(i) $\lim_{k\to\infty} d(s_{m(k)-1}, s_{n(k)+1}) = \varepsilon$;
(ii) $\lim_{k\to\infty} d(s_{m(k)}, s_{n(k)}) = \varepsilon$;
(iii) $\lim_{k\to\infty} d(s_{m(k)-1}, s_{n(k)}) = \varepsilon$;
(iv) $\lim_{k\to\infty} d(s_{m(k)+1}, s_{n(k)+1}) = \varepsilon$;
(v) $\lim_{k\to\infty} d(s_{m(k)}, s_{n(k)-1}) = \varepsilon$.

We observed that the proof of Theorem 3.2 (step 2) given in [11] is not correct. In this paper, we give its rigorous proof for a general case, that is, in the case of α–ψ-contractive mappings using the concept of C-functions. We support our obtained results by two examples and an application.

3 Main results

We introduce the following.

Definition 6. Let (M,d) be a RMS and α,η be given as in Definition 3. Such M is said to be α-regular with respect to η if for a sequence $\{u_n\}$ in M with $\alpha(u_n,u_{n+1}) \geqslant \eta(u_n,u_{n+1})$ for all $n \geqslant N$ and $u_n \to u$ as $n \to \infty$, then $\alpha(u_n,u) \geqslant \eta(u_n,u)$ for all $n \geqslant N$.

Our main result is

Theorem 1. *Let (M,d) be a complete Hausdorff RMS and $T: M \to M$ be an α-admissible mapping with respect to η. Suppose there exist $F \in C$ and $\psi,\phi \in \Psi$ such that, for $p,r \in M$,*

$$\alpha(p,r) \geqslant \eta(p,r) \implies \psi\big(d(Tp,Tr)\big) \leqslant F\big(\psi\big(m(p,r)\big),\phi\big(m(p,r)\big)\big), \qquad (1)$$

where

$$m(p,r) = \max\biggl\{d(p,r), d(p,Tp), d(r,Tr), \frac{d(p,Tp)d(r,Tr)}{1+d(p,r)}, \frac{d(p,Tp)d(r,Tr)}{1+d(Tp,Tr)}\biggr\}.$$

Assume that

(i) *there exists $u_0 \in M$ for which $\alpha(u_0,Tu_0) \geqslant \eta(u_0,Tu_0)$;*
(ii) *for all $u,v,w \in M$, $\alpha(u,v) \geqslant \eta(u,v)$ and $\alpha(v,w) \geqslant \eta(v,w)$ implies $\alpha(u,w) \geqslant \eta(u,w)$;*
(iii) *either T is continuous or M is α-regular with respect to η.*

Then there exists $x \in M$ such that $T^n x = x$ for some $n \in \mathbb{N}$, i.e., x is a periodic point. If in addition, $\alpha(x,Tx) \geqslant \eta(x,Tx)$ for each periodic point x, then T has a fixed point.

Proof. Given $u_0 \in M$ such that

$$\alpha(u_0,Tu_0) \geqslant \eta(u_0,Tu_0). \qquad (2)$$

Define a sequence $\{u_i\}$ in M by $u_i = Tu_{i-1} = T^i u_0$ for $i = 1,2,3,\dots$. If $u_{i_0+1} = u_{i_0}$ for some $i_0 \in \mathbb{N}$, then $Tu_{i_0} = u_{i_0}$.

From now on, suppose that $u_{i+1} \neq u_i$ for all $i \in \mathbb{N}$. Using (2) and the fact that T is an α-admissible mapping with respect to η, we have

$$\alpha(u_1,u_2) = \alpha\big(Tu_0,T^2u_0\big) \geqslant \eta\big(Tu_0,T^2u_0\big) = \eta(u_1,u_2).$$

By induction we get

$$\alpha(u_i,u_{i+1}) \geqslant \eta(u_i,u_{i+1}) \quad \text{for } i = 1,2,3,\dots .$$

Step 1. We will show that $\{d(u_i,u_{i+1})\}$ is nonincreasing and $d(u_i,u_{i+1}) \to 0$ as $i \to \infty$.

From (1) mention that

$$\psi\big(d(u_i,u_{i+1})\big) = \psi\big(d(Tu_{i-1},Tu_i)\big) \\ \leqslant F\big(\psi\big(m(u_{i-1},u_i)\big),\phi\big(m(u_{i-1},u_i)\big)\big), \tag{3}$$

where

$$m(u_{i-1},u_i) = \max\bigg\{d(u_{i-1},u_i),d(u_{i-1},Tu_{i-1}),d(u_i,Tu_i), \\ \frac{d(u_{i-1},Tu_{i-1})d(u_i,Tu_i)}{1+d(u_{i-1},u_i)},\frac{d(u_{i-1},Tu_{i-1})d(u_i,Tu_i)}{1+d(Tu_{i-1},Tu_i)}\bigg\} \\ = \max\big\{d(u_{i-1},u_i),d(u_i,u_{i+1})\big\}.$$

If $m(u_{i-1},u_i) = d(u_i,u_{i+1})$ for some $i \in \mathbb{N}$, then from (3),

$$\psi\big(d(u_i,u_{i+1})\big) \leqslant F\big(\psi\big(d(u_i,u_{i+1})\big),\phi\big(d(u_i,u_{i+1})\big)\big) \\ \leqslant \psi\big(d(u_i,u_{i+1})\big).$$

By Definition 4, $\psi(d(u_i,u_{i+1})) = 0$ or $\phi(d(u_i,u_{i+1})) = 0$. So $d(u_i,u_{i+1}) = 0$, which is a contradiction. Consequently, $m(u_{i-1},u_i) = d(u_{i-1},u_i)$ for every $i \in \mathbb{N}$. By (3) we get

$$\psi\big(d(u_i,u_{i+1})\big) \leqslant F\big(\psi\big(d(u_{i-1},u_i)\big),\phi\big(d(u_{i-1},u_i)\big)\big) \\ \leqslant \psi\big(d(u_{i-1},u_i)\big).$$

As ψ is nondecreasing,

$$d(u_i,u_{i+1}) \leqslant d(u_{i-1},u_i).$$

Hence, $\{d(u_i,u_{i+1})\}$ is a nonincreasing sequence of positive real numbers, so there exists $t \geqslant 0$ such that the limit

$$\lim_{i\to\infty} d(u_i,u_{i+1}) = t.$$

Also,

$$\lim_{i\to\infty} m(u_{i-1},u_i) = t.$$

As F, ψ and ϕ are continuous, we have

$$\lim_{i\to\infty}\psi\big(d(u_i,u_{i+1})\big) \leqslant \lim_{i\to\infty} F\big(\psi\big(m(u_{i-1},u_i)\big),\phi\big(m(u_{i-1},u_i)\big)\big) \\ = F\Big(\lim_{i\to\infty}\psi\big(m(u_{i-1},u_i),\lim_{i\to\infty}\phi\big(m(u_{i-1},u_i)\big)\big)\Big).$$

Therefore,

$$\psi(t) \leqslant F\big(\psi(t),\phi(t)\big) \leqslant \psi(t).$$

Again, by Definition 4 we get $t = 0$, that is,

$$\lim_{i\to\infty} d(u_i,u_{i+1}) = 0.$$

Step 2. We shall prove that $d(u_i, u_{i+2}) \to 0$ as $i \to \infty$.
By (1) we have

$$\begin{aligned}\psi\bigl(d(u_i, u_{i+2})\bigr) &\leqslant F\bigl(\psi\bigl(m(u_{i-1}, u_{i+1})\bigr), \phi\bigl(m(u_{i-1}, u_{i+1})\bigr)\bigr)\\ &\leqslant \psi\bigl(m(u_{i-1}, u_{i+1})\bigr).\end{aligned} \tag{4}$$

Thus,

$$\psi\bigl(d(u_i, u_{i+2})\bigr) \leqslant \psi\bigl(m(u_{i-1}, u_{i+1})\bigr).$$

Since ψ is an altering distance, we have

$$d(u_i, u_{i+2}) \leqslant m(u_{i-1}, u_{i+1}).$$

Note that

$$\begin{aligned}d(u_i, u_{i+2}) &\leqslant m(u_{i-1}, u_{i+1})\\ &= \max\biggl\{d(u_{i-1}, u_{i+1}), d(u_{i-1}, u_i), d(u_{i+1}, u_{i+2}),\\ &\qquad \frac{d(u_{i-1}, u_i)d(u_{i+1}, u_{i+2})}{1 + d(u_{i-1}, u_{i+1})}, \frac{d(u_{i-1}, u_i)d(u_{i+1}, u_{i+2})}{1 + d(u_i, u_{i+2})}\biggr\}\\ &\leqslant \max\bigl\{d(u_{i-1}, u_{i+1}), d(u_{i-1}, u_i), d(u_{i+1}, u_{i+2}),\\ &\qquad d(u_{i-1}, u_i)d(u_{i+1}, u_{i+2}), d(u_{i-1}, u_i)d(u_{i+1}, u_{i+2})\bigr\}\\ &\leqslant \max\bigl\{d(u_{i-1}, u_i) + d(u_i, u_{i+2}) + d(u_{i+2}, u_{i+1}),\\ &\qquad d(u_{i-1}, u_i)d(u_{i+1}, u_{i+2}), d(u_{i-1}, u_i)d(u_{i+1}, u_{i+2})\bigr\}.\end{aligned}$$

We shall see that $\lim_{i\to\infty} m(u_{i-1}, u_{i+1}) = \lim_{i\to\infty} d(u_i, u_{i+2})$. Using (4) and letting $i \to \infty$, we get

$$\begin{aligned}\psi\Bigl(\lim_{i\to\infty} d(u_i, u_{i+2})\Bigr) &\leqslant F\Bigl(\psi\Bigl(\lim_{i\to\infty} d(u_i, u_{i+2})\Bigr), \phi\Bigl(\lim_{i\to\infty} d(u_i, u_{i+2})\Bigr)\Bigr)\\ &\leqslant \psi\Bigl(\lim_{i\to\infty} d(u_i, u_{i+2})\Bigr).\end{aligned}$$

Thus,

$$F\Bigl(\psi\Bigl(\lim_{i\to\infty} d(u_i, u_{i+2})\Bigr), \phi\Bigl(\lim_{i\to\infty} d(u_i, u_{i+2})\Bigr)\Bigr) = \psi\Bigl(\lim_{i\to\infty} d(u_i, u_{i+2})\Bigr).$$

So $\psi(\lim_{i\to\infty} d(u_i, u_{i+2}) = 0$, or $\phi(\lim_{i\to\infty} d(u_i, u_{i+2}) = 0$. Again, by Definition 4 we deduce that

$$d(u_i, u_{i+2}) \to 0 \quad \text{as } i \to \infty.$$

Step 3. We shall show that T has a periodic point, i.e., $T^n h = h$ for some $h \in M$ and some $n \in \mathbb{N}$.

If T has no periodic point, then $u_i \neq u_j$ for all $j \neq i$. We will show that $\{u_i\}$ is a Cauchy sequence. We argue by contradiction. If $\{u_i\}$ is not a Cauchy sequence, then by Lemma 1 there exists $\epsilon > 0$ such that we can find subsequences $\{u_{i(k)}\}$ and $\{u_{j(k)}\}$

of $\{u_i\}$ with $i(k) > j(k) > k$ such that

$$\lim_{k\to\infty} d(u_{j(k)}, u_{i(k)}) = \lim_{k\to\infty} d(u_{j(k)-1}, u_{i(k)-1}) = \epsilon.$$

We have

$$\begin{aligned}
&m(u_{j(k)-1}, u_{i(k)-1})\\
&\quad = \max\biggl\{ d(u_{j(k)-1}, u_{i(k)-1}), d(u_{j(k)-1}, u_{j(k)}), d(u_{i(k)-1}, u_{i(k)}),\\
&\qquad \frac{d(u_{j(k)-1}, u_{j(k)})d(u_{i(k)-1}, u_{i(k)})}{1 + d(u_{j(k)-1}, u_{i(k)-1})}, \frac{d(u_{j(k)-1}, u_{j(k)})d(u_{i(k)-1}, u_{i(k)})}{1 + d(u_{j(k)}, u_{i(k)})} \biggr\}.
\end{aligned}$$

Then

$$\lim_{k\to\infty} m(u_{j(k)-1}, u_{i(k)-1}) = \epsilon.$$

By condition (ii) we have $\alpha(u_{j(k)-1}, u_{i(k)-1}) \geqslant \eta(u_{j(k)-1}, u_{i(k)-1})$. Therefore, by (1),

$$\begin{aligned}
&\psi\bigl(d(Tu_{j(k)-1}, Tu_{i(k)-1})\bigr)\\
&\quad \leqslant F\bigl(\psi\bigl(m(u_{j(k)-1}, u_{i(k)-1})\bigr), \phi\bigl(m(u_{j(k)-1}, u_{i(k)-1})\bigr)\bigr).
\end{aligned}$$

As F, ψ and ϕ are continuous, so as $k \to \infty$,

$$\psi(\epsilon) \leqslant F\bigl(\psi(\epsilon), \phi(\epsilon)\bigr) \leqslant \psi(\epsilon).$$

Thus, $\psi(\epsilon) = 0$ or $\phi(\epsilon) = 0$. Hence, $\epsilon = 0$, a contradiction. We deduce that $\{u_i\}$ is a (g.m.s) Cauchy sequence. Then there exists $b \in M$ (since M is complete) such that $\{u_i\}$ is (g.m.s) convergent to b.

In the case that T is continuous, we have

$$u_{i+1} = Tu_i \to Tb \quad \text{as } i \to \infty.$$

Since M is Hausdorff, we get that $b = Tb$. So T has a periodic point.

Suppose now that M is α-regular with respect to η. From condition (iii) $\alpha(u_i, b) \geqslant \eta(u_i, b)$ for all $i \in \mathbb{N}$. This implies that

$$\psi(Tu_i, Tb) \leqslant F\bigl(\psi\bigl(m(u_i, b)\bigr), \phi\bigl(m(u_i, b)\bigr)\bigr), \tag{5}$$

where

$$\begin{aligned}
m(u_i, b) = \max\biggl\{ & d(u_i, b), d(u_i, u_{i+1}), d(b, Tu),\\
& \frac{d(b, Tb)d(u_i, u_{i+1})}{1 + d(u_i, x)}, \frac{d(b, Tb)d(u_i, u_{i+1})}{1 + d(u_{i+1}, Tb)} \biggr\}.
\end{aligned} \tag{6}$$

Since $\{u_i\} \to b$ as $i \to \infty$,

$$\lim_{i\to\infty} m(u_i, x) = d(b, Tb).$$

Taking limit as $i \to \infty$ in (5),

$$\psi\bigl(d(b, Tb)\bigr) \leqslant F\bigl(\psi\bigl(d(b, Tb)\bigr), \phi\bigl(d(b, Tb)\bigr)\bigr) \leqslant \psi\bigl(d(b, Tb)\bigr).$$

Consequently, $\psi(d(b,Tb)) = 0$ or $\phi(d(b,Tb)) = 0$, so $d(b,Tb) = 0$, i.e., $b = Tb$. Hence, T has a periodic point.

Step 4. We claim that T has a fixed point.

From Step 3 there exists z in M such that $z = T^k z$. Clearly, z is a fixed point T in the case that $k = 1$. We will prove that $y = T^{k-1}z$ is a fixed point of T in case that $k > 1$. If possible, let $T^{k-1}z \neq T^k z$ for all $k > 1$. Also, $\alpha(z,Tz) \geqslant \eta(z,Tz)$ for a periodic point z. Thus, from the contractive inequality (1) we have

$$\begin{aligned}&\psi\bigl(d\bigl(T^{k-1}z,T^kz\bigr)\bigr)\\&\quad\leqslant F\bigl(\psi\bigl(m\bigl(T^{k-2}z,T^{k-1}z\bigr)\bigr),\phi\bigl(m\bigl(T^{k-2}z,T^{k-1}z\bigr)\bigr)\bigr),\end{aligned}\tag{7}$$

where

$$\begin{aligned}&m\bigl(T^{k-2}z,T^{k-1}z\bigr)\\&\quad=\max\biggl\{d(T^{k-2}z,T^{k-1}z),d(T^{k-2}z,T^{k-1}z),d(T^{k-1}z,T^kz),\\&\qquad\frac{d(T^{k-2}z,T^{k-1}z)d(T^{k-1}z,T^kz)}{1+d(T^{k-2}z,T^{k-1}z)},\frac{d(T^{k-2}z,T^{k-1}z)d(T^{k-1}z,T^kz)}{1+d(T^{k-1}z,T^kz)}\biggr\}\\&\quad=\max\bigl\{d\bigl(T^{k-2}z,T^{k-1}z\bigr),d\bigl(T^{k-1}z,T^kz\bigr)\bigr\}.\end{aligned}$$

If for some k, $m(T^{k-2}z,T^{k-1}z) = d(T^{k-1}z,T^kz)$, then from (7) we get

$$\begin{aligned}\psi\bigl(d\bigl(T^{k-1}z,T^kz\bigr)\bigr)&\leqslant F\bigl(\psi\bigl(d\bigl(T^{k-1}z,T^kz\bigr)\bigr),\phi\bigl(d\bigl(T^{k-1}z,T^kz\bigr)\bigr)\bigr)\\&\leqslant\psi\bigl(d\bigl(T^{k-1}z,T^kz\bigr)\bigr).\end{aligned}$$

Hence, $\psi(d(T^{k-1}z,T^kz)) = 0$ or $\phi(d(T^{k-1}z,T^kz)) = 0$. That is, $d(T^{k-1}z,T^kz) = 0$, which is a contradiction. Thus, $m(T^{k-2}z,T^{k-1}z) = d(T^{k-2}z,T^{k-1}z)$ for every k. Now, as T is α-admissible with respect to η and $\alpha(z,Tz) \geqslant \eta(z,Tz)$, we have

$$\begin{aligned}\psi\bigl(d\bigl(T^{k-1}z,T^kz\bigr)\bigr)&\leqslant F\bigl(\psi\bigl(d\bigl(T^{k-2}z,T^{k-1}z\bigr)\bigr),\phi\bigl(d\bigl(T^{k-2}z,T^{k-1}z\bigr)\bigr)\bigr)\\&\leqslant\psi\bigl(d\bigl(T^{k-2}z,T^{k-1}z\bigr)\bigr).\end{aligned}$$

Therefore, $\{d(T^{k-1}z,T^kz)\}$ is a nonincreasing sequence of nonnegative reals. We have

$$\begin{aligned}\psi\bigl(d(z,Tz)\bigr)&=\psi\bigl(d\bigl(T^kz,T^{k+1}z\bigr)\bigr)\leqslant\psi\bigl(d\bigl(T^{k-1}z,T^kz\bigr)\bigr)\\&\leqslant F\bigl(\psi\bigl(d\bigl(T^{k-2}z,T^{k-1}z\bigr)\bigr),\phi\bigl(d\bigl(T^{k-2}z,T^{k-1}z\bigr)\bigr)\bigr)\\&\leqslant\psi\bigl(d\bigl(T^{k-2}z,T^{k-1}z\bigr)\bigr)\leqslant\cdots\leqslant F\bigl(\psi\bigl(d(z,Tz),\phi\bigl(d(z,Tz)\bigr)\bigr)\\&\leqslant\psi\bigl(d(z,Tz)\bigr).\end{aligned}$$

Consequently, $\psi(d(T^{k-2}z,T^{k-1}z)) = 0$ or $\phi(d(T^{k-2}z,T^{k-1}z)) = 0$, which implies that $d(T^{k-2}z,T^{k-1}z) = 0$, and therefore, $T^{k-2}z = T^{k-1}z$, that is, $T^{k-1}z = T^k z$, which is a contradiction. Hence, the assumption that $y = T^{k-1}z$ is not a fixed point of T, is not true. Consequently, T has a fixed point. □

Theorem 2. *To ensure uniqueness of the fixed point in Theorem* 1*, we add the following condition*: *For all* $a, b \in F(T) = \{w \in M\colon Tw = w\}$*,* $\alpha(a,b) \geqslant \eta(a,b)$*.*

Proof. Let us assume that $y_1, y_2 \in M$ are two distinct fixed points of T. Using the inequality $\alpha(y_1,y_2) \geqslant \eta(y_1,y_2)$ and (1), one writes

$$\psi\bigl(d(y_1,y_2)\bigr) = \psi\bigl(d(Ty_1,Ty_2)\bigr) \leqslant F\bigl(\psi\bigl(m(y_1,y_2)\bigr), \phi\bigl(m(y_1,y_2)\bigr)\bigr), \tag{8}$$

where

$$\begin{aligned} m(y_1,y_2) = \max\biggl\{ & d(y_1,y_2), d(y_1,Ty_1), d(y_2,Ty_2), \\ & \frac{d(y_1,Ty_1)d(y_2,Ty_2)}{1+d(y_1,y_2)}, \frac{d(y_1,Ty_1)d(y_2,Ty_2)}{1+d(Ty_1,Ty_2)} \biggr\} \\ = d(y_1,y_2). \end{aligned}$$

Returning to (8), we see that

$$\psi\bigl(d(y_1,y_2)\bigr) \leqslant F\bigl(\psi\bigl(d(y_1,y_2)\bigr), \phi\bigl(d(y_1,y_2)\bigr)\bigr).$$

Thus, $\psi(d(y_1,y_2)) = 0$, or $\phi(d(y_1,y_2)) = 0$. This implies that $d(y_1,y_2) = 0$, that is, the fixed point is unique. $\square$

If we take $F(s,t) = s - t$, we have following corollary.

Corollary 1. *Let* (M,d) *be a Hausdorff and complete RMS, and let* $T : M \to M$ *be an* α*-admissible mapping with respect to* η*. Assume that there exist* $\psi, \phi \in \Psi$ *such that, for* $p, r \in M$*,*

$$\alpha(p,r) \geqslant \eta(p,r) \implies d(Tp,Tr) \leqslant \psi\bigl(m(p,r)\bigr) - \phi\bigl(m(p,r)\bigr),$$

where

$$\begin{aligned} m(p,r) = \max\biggl\{ & d(p,r), d(p,Tp), d(r,Tr), \\ & \frac{d(p,Tp)d(r,Tr)}{1+d(p,r)}, \frac{d(p,Tp)d(r,Tr)}{1+d(Tp,Tr)} \biggr\}. \end{aligned}$$

Suppose also that the following assertions hold:

(i) *there exists* $u_0 \in M$ *such that* $\alpha(u_0,Tu_0) \geqslant \eta(u_0,Tu_0)$*;*
(ii) *for all* $u, v, w \in M$*,* $\alpha(u,v) \geqslant \eta(u,v)$ *and* $\alpha(v,w) \geqslant \eta(v,w)$ *implies* $\alpha(u,w) \geqslant \eta(u,w)$*;*
(iii) *either* T *is continuous or* M *is* α*-regular with respect to* η*.*

Then T *has a periodic point* $x \in M$*. If in addition,* $\alpha(x,Tx) \geqslant \eta(x,Tx)$ *holds for each periodic point, then* T *has a fixed point. Moreover, if for all* $a, b \in F(T)$*, we have* $\alpha(a,b) \geqslant \eta(a,b)$*, then the fixed point is unique.*

Taking $\psi(t) = t$ in Corollary 1, we have the following.

Corollary 2. *Let (M,d) be a Hausdorff and complete RMS. Let $T:M\to M$ be an α-admissible mapping with respect to η. Assume there exists $\phi\in\Psi$ such that, for $p,r\in M$,*

$$\alpha(p,r)\geqslant\eta(p,r)\implies d(Tp,Tr)\leqslant m(p,r)-\phi\bigl(m(p,r)\bigr),$$

where

$$m(p,r)=\max\biggl\{d(p,r),d(p,Tp),d(r,Tr),\\ \frac{d(p,Tp)d(r,Tr)}{1+d(p,r)},\frac{d(p,Tp)d(r,Tr)}{1+d(Tp,Tr)}\biggr\}.$$

Suppose also that the following assertions hold:

(i) *there exists $u_0\in M$ such that $\alpha(u_0,Tu_0)\geqslant\eta(u_0,Tu_0)$;*
(ii) *for all $u,v,w\in M$, $\alpha(u,v)\geqslant\eta(u,v)$ and $\alpha(v,w)\geqslant\eta(v,w)$ implies $\alpha(u,w)\geqslant\eta(u,w)$;*
(iii) *either T is continuous or M is α-regular with respect to η.*

Then T has a periodic point $x\in M$. If in addition $\alpha(x,Tx)\geqslant\eta(x,Tx)$ for each periodic point, then T has a fixed point. Moreover, if for all $a,b\in F(T)$, we have $\alpha(a,b)\geqslant\eta(a,b)$, then the fixed point is unique.

Consider $\phi(t)=(1-q)(t)$ for $0<q<1$, in Corollary 2.

Corollary 3. *Let (M,d) be a Hausdorff and complete RMS. Let $T:M\to M$ be an α-admissible mapping with respect to η such that, for $p,r\in M$,*

$$\alpha(p,r)\geqslant\eta(p,r)\implies d(Tp,Tr)\leqslant qm(p,r),$$

where $m(p,r)$ is the same as in Corollary 2. Suppose also that the following holds:

(i) *there exists $u_0\in M$ such that $\alpha(u_0,Tu_0)\geqslant\eta(u_0,Tu_0)$;*
(ii) *for all $u,v,w\in M$, $\alpha(u,v)\geqslant\eta(u,v)$ and $\alpha(v,w)\geqslant\eta(v,w)$ implies $\alpha(u,w)\geqslant\eta(u,w)$;*
(iii) *either T is continuous or M is α-regular with respect to η.*

Then T has a periodic point $x\in M$. If in addition, $\alpha(x,Tx)\geqslant\eta(x,Tx)$ for each periodic point, then T has a fixed point. Moreover, if for all $a,b\in F(T)$, we have $\alpha(a,b)\geqslant\eta(a,b)$, then the fixed point is unique.

4 Examples

We present the following examples.

Example 2. Consider $M=[0,1]$ and define $T:M\to M$ as

$$Tp=\begin{cases}crp+\frac{1}{2}, & 0\leqslant p<\frac{1}{2},\\ \frac{1}{2}, & \frac{1}{2}\leqslant p\leqslant 1.\end{cases}$$

Define $\alpha, \eta : M \times M \to [0, \infty)$ as $\alpha(p, r) = 3$ and $eta(p, r) = 2$ for all $p, r \in M$, respectively. Further, let $d : M \times M \to [0, \infty)$ be given as $d(p, r) = d(r, p)$, $d(p, r) = 0$ if and only if $p = r$ and

$$d(p,r) = \begin{cases} \frac{3}{5}, & 0 \leqslant p, r \leqslant \frac{1}{2}, \\ \frac{3}{20}, & \frac{1}{2} \leqslant p, r \leqslant 1, \\ \frac{1}{3}, & 0 \leqslant p \leqslant \frac{1}{2},\ \frac{1}{2} \leqslant r \leqslant 1. \end{cases}$$

Further, it can be easily checked that

(i) for $0 \leqslant p, r \leqslant 1/2$, we have $d(Tp, Tr) = 3/20$ and $m(p, r) = 3/5$;
(ii) for $1/2 \leqslant p, r \leqslant 1$, we have $d(Tp, Tr) = 0$ and $m(p, r) = 3/20$;
(iii) for $0 \leqslant p \leqslant 1/2$, $1/2 \leqslant r \leqslant 1$, we have $d(Tp, Tr) = 3/20$ and $m(p, r) = 1/3$.

Consider the functions $F : [0, \infty) \times [0, \infty) \to \mathbb{R}$ and $\phi,\ \psi : [0, \infty) \to [0, \infty)$ defined as $F(s, t) = s - t$, $\psi(t) = 4t/5$ and $\phi(t) = t/3$, respectively. Then all the conditions of Theorem 1 (and Theorem 2) hold, and hence, T has a unique fixed point, which is $z = 1/2$.

In the second example, the rectangular metric space (M, d) is not Hausdorff, and the mapping T has no fixed point. So the hypothesis that (M, d) is Hausdorff is capital to ensure the existence of a fixed point.

Example 3. Let $M_1 = \{0, 2\}$, $M_2 = \{1, 1/2, 1/3, \dots\}$ and $M = M_1 \cup M_2$. Define $d : M \times M \to [0, \infty)$ as follows:

$$d(p,r) = \begin{cases} 0, & p = r, \\ 1, & p \neq r \text{ and } \{p, r\} \subset M_1 \text{ or } \{p, r\} \subset M_2, \\ r, & p \in M_1 \text{ and } r \in M_2, \\ p, & p \in M_2 \text{ and } r \in M_1. \end{cases}$$

Then (M, d) is a complete Rectangular metric space. Note that (M, d) is not Hausdorff because there is no $s > 0$ such that $B(0, s) \cap B(2, s) = \emptyset$. Given $\alpha, \eta : M \times M \to [0, \infty)$ as

$$\alpha(p,r) = \begin{cases} 4, & p \neq 0 \text{ or } r \neq \frac{1}{j}, \\ 2, & p = 0 \text{ and } r = \frac{1}{j}, \end{cases} \qquad \eta(p,r) = 3, \quad p, r \in M \times M.$$

Define $T : M \to M$ by

$$T(0) = \frac{1}{2}, \quad T(2) = 0 \quad \text{and} \quad T\left(\frac{1}{j}\right) = 0 \quad \text{for } \frac{1}{j} \in M_2.$$

For our convenience, we use following symbols:

$$A_1 = d(p, r), \qquad A_2 = d(p, Tp), \qquad A_3 = d(r, Tr),$$
$$A_4 = \frac{d(p, Tp)d(r, Tr)}{1 + d(p, r)} \quad \text{and} \quad A_5 = \frac{d(p, Tp)d(r, Tr)}{1 + d(Tp, Tr)}.$$

We have

	$Tp=1/2,\ Tr=0$		$Tp=0,\ Tr=0$	
	$p=0$ $r=2$	$p=0$ $r=1/j$	$p=2$ $r=1/j$	$p=1/j$ $r=1/i,\ i\neq j$
A_1	1	$1/j$	$1/j$	1
A_2	0.5	0.5	1	$1/j$
A_3	1	$1/j$	$1/j$	$1/i$
A_4	0.25	$1/2(j+1)$	$1/j+1$	$1/2ij$
A_5	0.33	$1/3j$	$1/j$	$1/ij$
$d(Tp,Tr)$	0.5	0.5	0	0
$m(p,r)$	1	1 if $j=1$, 0.5 if $j\geqslant 2$	1	1

Define the functions $F:[0,\infty)\times[0,\infty)\to\mathbb{R}$ and ψ, $\phi:[0,\infty)\to[0,\infty)$ as $F(s,t)=5s/6$, $\psi(t)=2t/3$ and $\phi(t)=t/4$, respectively. From the above table it can be easily seen that

$$\psi\bigl(d(Tp,Tr)\bigr)\leqslant F\bigl(\psi\bigl(m(p,r)\bigr),\phi\bigl(m(p,r)\bigr)\bigr)$$

holds whenever $\alpha(p,r)\geqslant\eta(p,r)$, but $F(T)$ is empty.

5 Application

Fractional calculus has recently been of great interest because of both the intensive development of the theory of fractional calculus itself and the applications of such constructions in various sciences, engineering etc. For details, see the monographs of Miller and Ross [29], Podlubny [30], Yang [39] and the papers [9, 12, 13, 16, 19–22, 27, 31, 38, 40–42].

In principle, one may reduce differential equation to an integral equation and apply to it basic technique of nonlinear analysis (fixed point theorems). In this section, we apply the fixed point result derived in Corollary 3 to guarantee the existence of a solution of a fractional-order functional differential equation with infinite delay.

Consider the initial value problem

$$\begin{aligned}&D^{\beta}y(t)=g(t,y_t),\quad t\in S=[0,a],\ 0<\beta<1,\\&y(t)=\psi(t),\quad -\infty<t\leqslant 0,\end{aligned}\tag{9}$$

and D^{β} is the Riemann–Liouville fractional derivative, $g:S\times P\to\mathbb{R}$, $\psi\in P$ is such that $\psi(0)=0$, and P is known as a phase space (state space) satisfying the following axioms, which were introduced in [18] by Hale and Kato (see also [7, 26]).

(H1) If $y:(-\infty,a]\to\mathbb{R}$ and $y_0\in P$, then for every $t\in[0,a]$,

(i) $y_t\in P$;

(ii) $\|y_t\|_P\leqslant K_1(t)\sup\{|y(w)|,\ w\in[0,s]\}+K_2(t)\|y_0\|_P$;

(iii) $|y(t)|\leqslant K_3\|y_t\|_P$, where K_3 is a nonnegative constant, $K_1:S\to[0,\infty)$ is continuous, $K_2:[0,\infty)\to[0,\infty)$ is locally bounded and K_1, K_2, K_3 do not depend on $y(t)$;

(H2) For $y(t) \in (H_1)$, y_t is a P-valued continuous function on S;
(H3) P is a complete space.

Let $y_t \in P$ be defined as

$$y_t(x) = y(t + x), \quad -\infty < x \leqslant 0,$$

where y is a function defined on P. Mention that $y_t(\cdot)$ represents the history of the state from $-\infty$ up to present time t. Consider the Banach space $C(S, \mathbb{R})$ of all continuous functions from S into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(s)|, \, s \in S\}.$$

Denote $\Delta = \{y : (-\infty, a] \to \mathbb{R}\colon y|_{(-\infty,0]} \in P,\ y|_{[0,a]} \text{ is continuous}\}$. Thus, $y \in \Delta$ is a solution of (9) if y satisfies $D^\beta y(t) = g(t, y_t)$ on S with the condition $y(t) = \psi(t)$ on $(-\infty, 0]$.

Before proving our existence theorem for problem (9), we state the following auxiliary lemma.

Lemma 2. (*See* [15].) *Let $\beta \in (0, 1)$, and let $r : (0, a] \to \mathbb{R}$ be continuous. Then y is a solution of the fractional integral equation*

$$y(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t - v)^{\beta - 1} r(v) \, \mathrm{d}v$$

if and only if y is a solution of the initial value problem for the fractional differential equation

$$D^\beta y(t) = r(t), \quad t \in (0, a],$$
$$y(0) = 0.$$

Now, consider the operator $\mathcal{M} : \Delta \to \Delta$ defined by

$$\mathcal{M}(y)(t) = \begin{cases} \psi(t), & t \in (-\infty, 0], \\ \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1} g(s, y_s) \, \mathrm{d}s, & t \in [0, a]. \end{cases}$$

Define $x(\cdot) : (-\infty, a] \to \mathbb{R}$ by

$$x(t) = \begin{cases} \psi(t), & t \in (-\infty, 0], \\ 0, & t \in [0, a]. \end{cases}$$

Then $x_0 = \psi$. For each $z \in C([0, a], \mathbb{R})$ with $z(0) = 0$, we denote

$$\bar{z}(t) = \begin{cases} 0, & t \in (-\infty, 0], \\ z(t), & t \in [0, a]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation

$$y(t) = \frac{1}{\Gamma(\beta)} \int\limits_0^t (t-s)^{\beta-1} g(s, y_s)\,\mathrm{d}s,$$

we can decompose $y(\cdot)$ as $y(t) = \bar{z}(t) + x(t)$, $0 \leqslant t \leqslant a$, which implies $y_t = \bar{z}_t + x_t$ for every $0 \leqslant t \leqslant a$, and the function $z(\cdot)$ satisfies

$$z(t) = \frac{1}{\Gamma(\beta)} \int\limits_0^t (t-s)^{\beta-1} g(s, \bar{z}_s + x_s)\,\mathrm{d}s.$$

Set

$$C_0 = \bigl\{ z \in C\bigl([0,a], \mathbb{R}\bigr) \colon z_0 = 0 \bigr\},$$

and let $\|\cdot\|_a$ be the seminorm in C_0 defined by

$$\begin{aligned} \|z\|_a &= \|z_0\|_P + \sup\bigl\{ \bigl|z(t)\bigr|, \, 0 \leqslant t \leqslant a \bigr\} \\ &= \sup\bigl\{ \bigl|z(t)\bigr|, \, 0 \leqslant t \leqslant a \bigr\}, \quad z \in C_0. \end{aligned}$$

Mention that C_0 is a Banach space with the norm $\|\cdot\|_a$. Here, we observe that

$$C_0 = \Delta = P, \quad \in [0,a].$$

Now, we are ready to prove the following existence result.

Theorem 3. *Let $f : S \times C_0 \to \mathbb{R}$ and $\zeta : C_0 \times C_0 \to \mathbb{R}$ be given functions. Assume that*

(i) *there exists $c > 0$ such that, for every $u_1, u_2 \in P$,*

$$\bigl| g(t, u_1) - g(t, u_2) \bigr| \leqslant c \|u_1 - u_2\|_P, \quad t \in S,$$

and $a^\beta K_a c / \Gamma(\beta+1) = \lambda < 1$, where $K_a = \sup\{|K(t)|, \, t \in S\}$;

(ii) *for all $t \in S = [0,a]$, $\zeta(x(t), y(t)) \geqslant 0$ implies $\zeta(Px(t), Py(t)) \geqslant 0$;*

(iii) *there exists $x_0 \in C_0$ such that $\zeta(x_0(t), Px_0(t)) \geqslant 0$ for all $t \in S$, where the operator $P : C_0 \to C_0$ is given by*

$$(Pz)(t) = \frac{1}{\Gamma(\beta)} \int\limits_0^t (t-s)^{\beta-1} g(s, \, \bar{z}_s + x_s)\,\mathrm{d}s, \quad t \in [0,b];$$

(iv) *if $\{x_n\}$ is a sequence in C_0 converging to a point $x \in C_0$ and $\zeta(x_n, x_{n+1}) \geqslant 0$ for all $n \in \mathbb{N}$, then $\zeta(x_n, x) \geqslant 0$ for all $n \in \mathbb{N}$.*

Then the initial value problem (9) *has a solution on the interval* $(-\infty, a]$.

Proof. It is clear that the operator $\mathcal{M}$ has a fixed point is equivalent to the fact that P has a fixed point. We will prove that P has a fixed point. Let $z, z^* \in C_0$ such that $\zeta(z(t), z^*(t)) \geqslant 0$ for all $t \in [0, a]$. By (i) we have

$$\begin{aligned}
&\bigl|P(z)(t) - P(z^*)(t)\bigr| \\
&\quad\leqslant \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \bigl|g(s,\, \bar{z}_s + x_s) - g(s,\, \bar{z}^*_s + x_s)\bigr| \,\mathrm{d}s \\
&\quad\leqslant \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} c \|\bar{z}_s - \bar{z}^*_s\|_P \,\mathrm{d}s \\
&\quad\leqslant \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} c K_a \sup_{s\in[o,t]} \bigl\|z(s) - z^*(s)\bigr\| \,\mathrm{d}s \\
&\quad\leqslant \frac{K_a}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} c \,\mathrm{d}s \,\|z - z^*\|_a.
\end{aligned}$$

Therefore,

$$\bigl\|P(z) - P(z^*)\bigr\|_a \leqslant \frac{c a^\beta K_a}{\Gamma(\beta+1)} \|z - z^*\|_a,$$

that is,

$$d\bigl(P(z), P(z^*)\bigr) \leqslant \lambda d(z, z^*) \leqslant \lambda m(z, z^*),$$

where

$$\begin{aligned}
m(z, z^*) = \max\biggl\{& d(z, z^*), d(z, Pz), d(z^*, Pz^*), \\
& \frac{d(z, Pz) d(z^*, Pz^*)}{1 + d(z, z^*)}, \frac{d(z, Pz) d(z^*, Pz^*)}{1 + d(Pz, Pz^*)}\biggr\}.
\end{aligned}$$

Now, we define $\alpha : C_0 \times C_0 \to [0, \infty)$ and $\eta : C_0 \times C_0 \to [0, \infty)$ as follows:

$$\alpha(x, y) = \begin{cases} 1, & \zeta(x(t), y(t)) \geqslant 0,\ t \in [0, a], \\ 0 & \text{otherwise}, \end{cases}$$

and $\eta(x, y) = 1$ for all $x, y \in C_0$. Now, by condition (ii) we get

$$\begin{aligned}
&\alpha(x, y) \geqslant \eta(x, y) \implies \zeta\bigl(x(t), y(t)\bigr) \geqslant 0 \implies \zeta\bigl(Px(t), Py(t)\bigr) \geqslant 0 \\
&\quad\implies \alpha(Px, Py) \geqslant \eta(Px, Py).
\end{aligned}$$

From (iii) there is $x_0 \in C_0$ such that $\alpha(x_0, Px_0) \geqslant \eta(x_0, Px_0)$. Further, from (iv), condition (iii) of Corollary 3 holds. Thus, all conditions of Corollary 3 are verified. Hence, P has unique fixed point. □

6 Conclusion

Taking into account its interesting applications, searching for fixed point theorems involving contractive-type conditions received considerable attention through the last few decades. In this connection, based on the new idea of $\phi - \psi$-contractive mappings satisfying an admissibility-type condition in generalized metric spaces using the concept of $\mathcal{C}$-functions, we studied fixed point results for such mappings and established an existence result for fractional-order functional differential equations with infinite delay. The new concepts lead to further investigations and applications. For instance, it will be interesting to apply these concepts in those metric spaces, which do not involve full form of triangle inequality such as b metrics spaces [14] and partial metric spaces [28].

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